Solving ill-conditionned linear systems using extended sparsification: an application to extruded meshes

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Direct methods
(Sparse) LU
+ Pre-Ordering (ND)

- Very Robust
- Very Accurate
- Generic
- (Very) Costly

Incomplete Factorizations
Incomplete LU / Sparsification methods / …

- Tunable Accuracy
- Tunable Cost
- (Fairly) Generic
- Direct method or Preconditioner

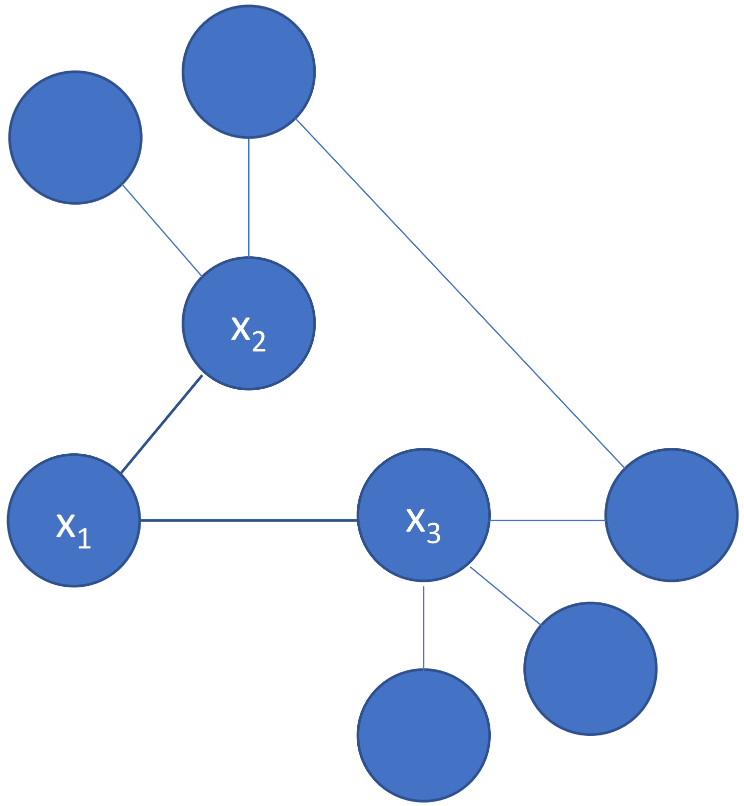
Iterative methods
CG / GMRES
+ Custom Preco

- Cheap
- (Very) Specific
- Complex Convergence
- Need Domain knowledge

Extended-Sparsification

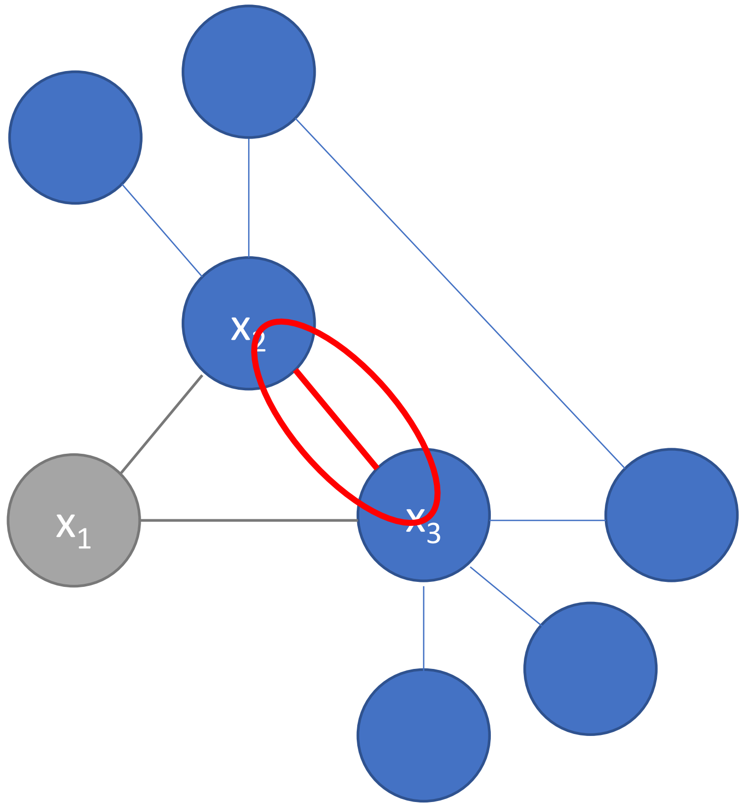
Basic Ideas

Gaussian elimination



$$\begin{bmatrix} A_{11} & A_{21}^T & A_{31}^T \\ A_{21} & A_{22} & \\ A_{31} & & A_{33} \end{bmatrix}$$

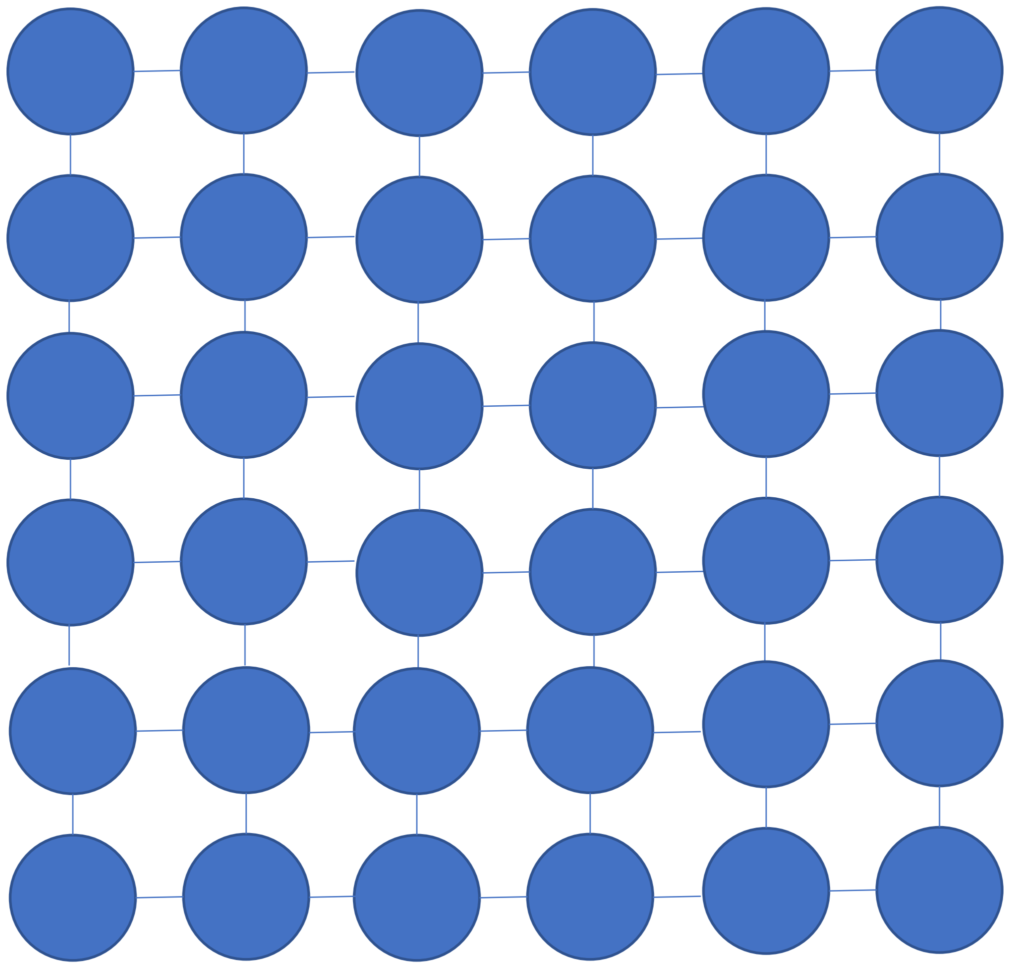
Gaussian elimination



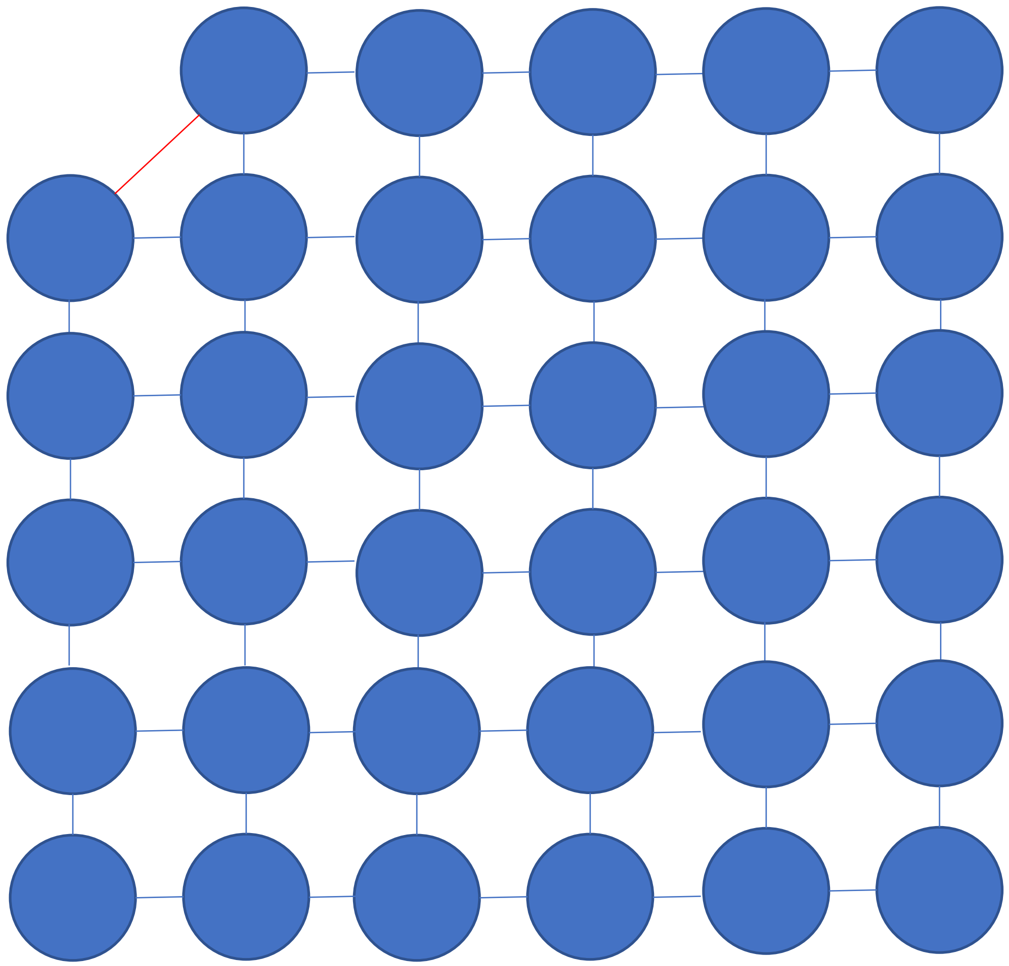
$$\begin{bmatrix} I & & \\ & A_{22} & D^T \\ & D & A_{33} \end{bmatrix}$$

$$D = -A_{31}A_{11}^{-1}A_{12}$$

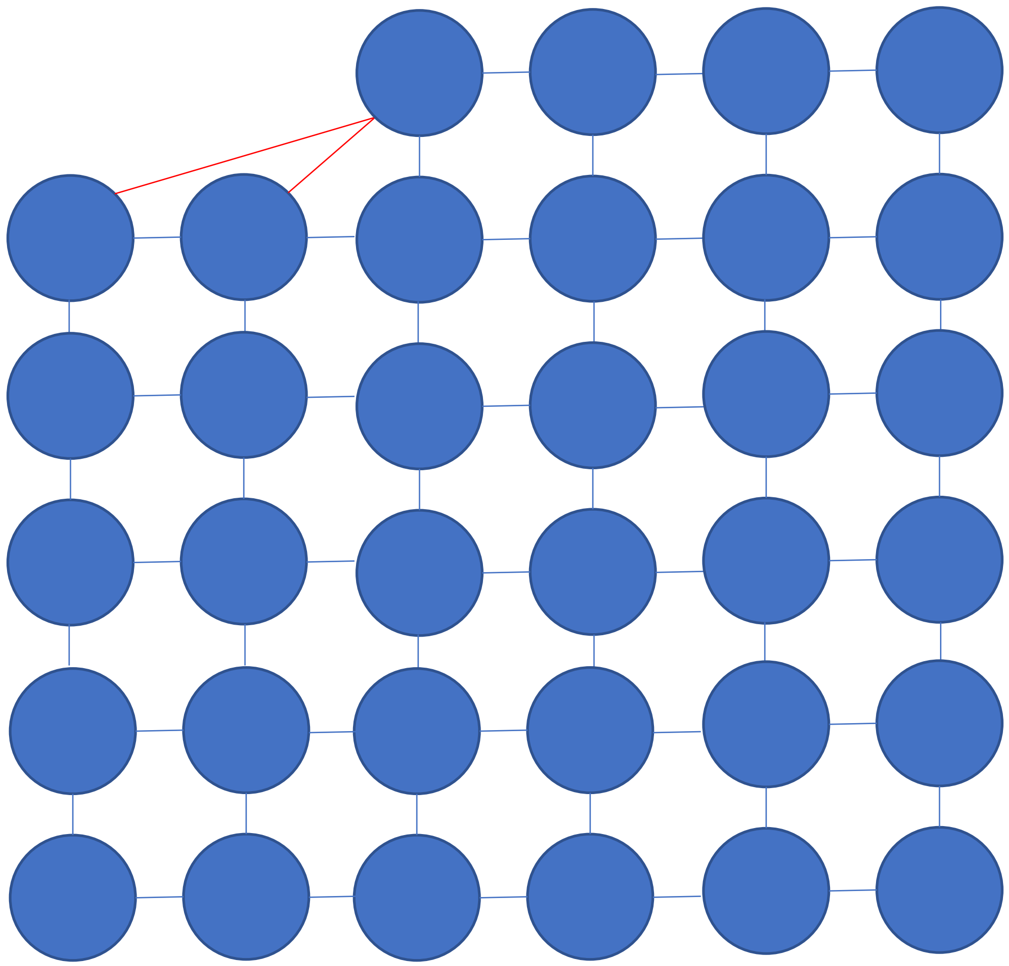
Everything fills!



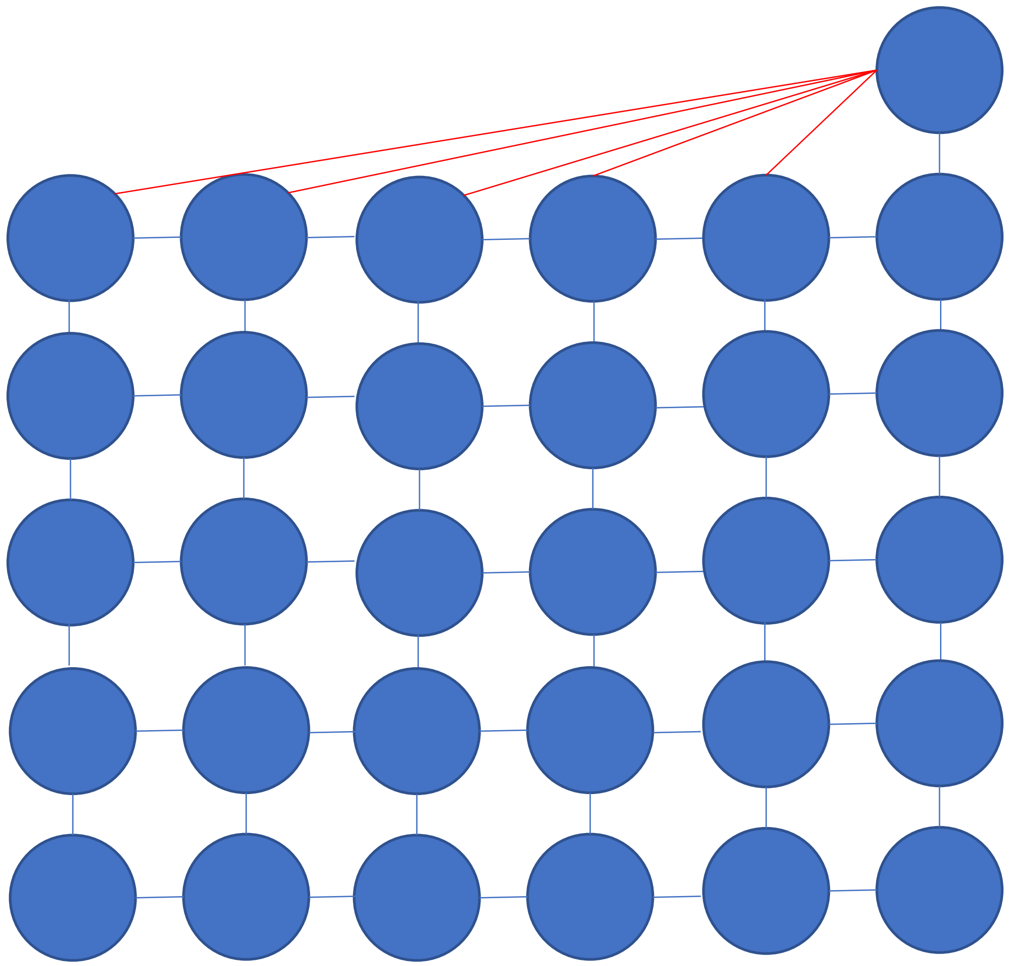
Everything fills!



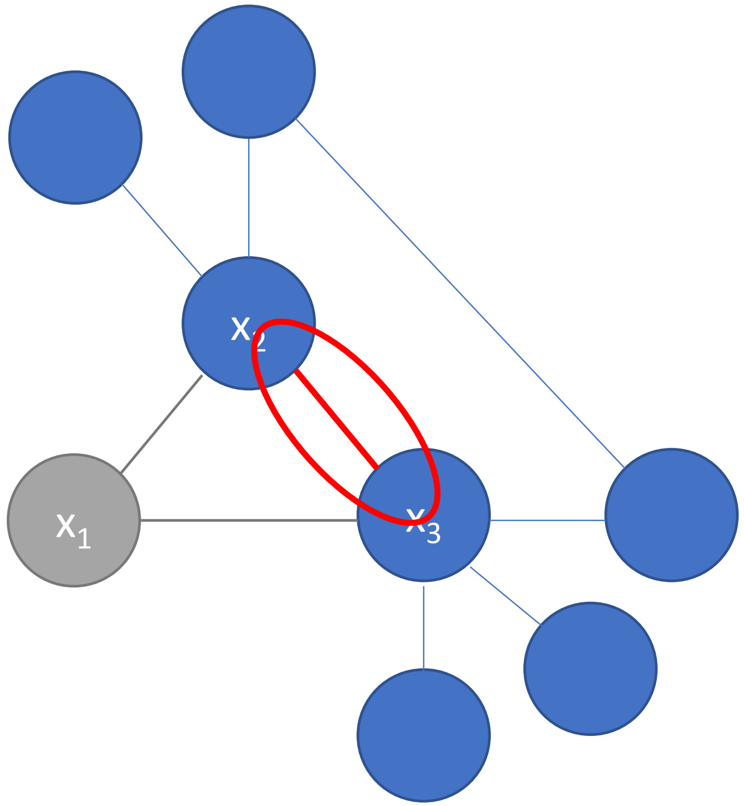
Everything fills!



Everything fills!



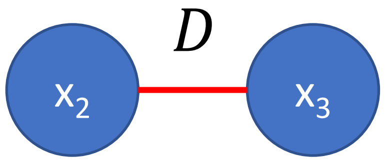
Solution? Low-Rank!



$$\begin{bmatrix} I & & \\ & A_{22} & D^T \\ & D & A_{33} \end{bmatrix}$$

$$D = -A_{31}A_{11}^{-1}A_{12} \approx UKV^T$$

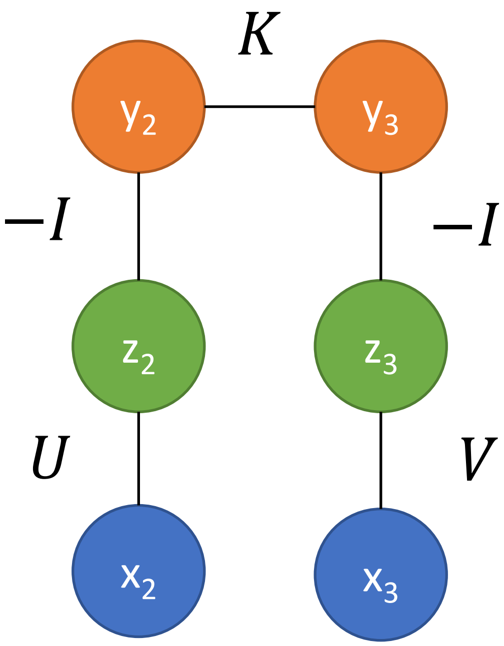
Solution? Low-Rank!



$$\begin{bmatrix} A_{22} & \color{red}{D} \\ \color{red}{D^\top} & A_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

$$D \approx UKV^T$$

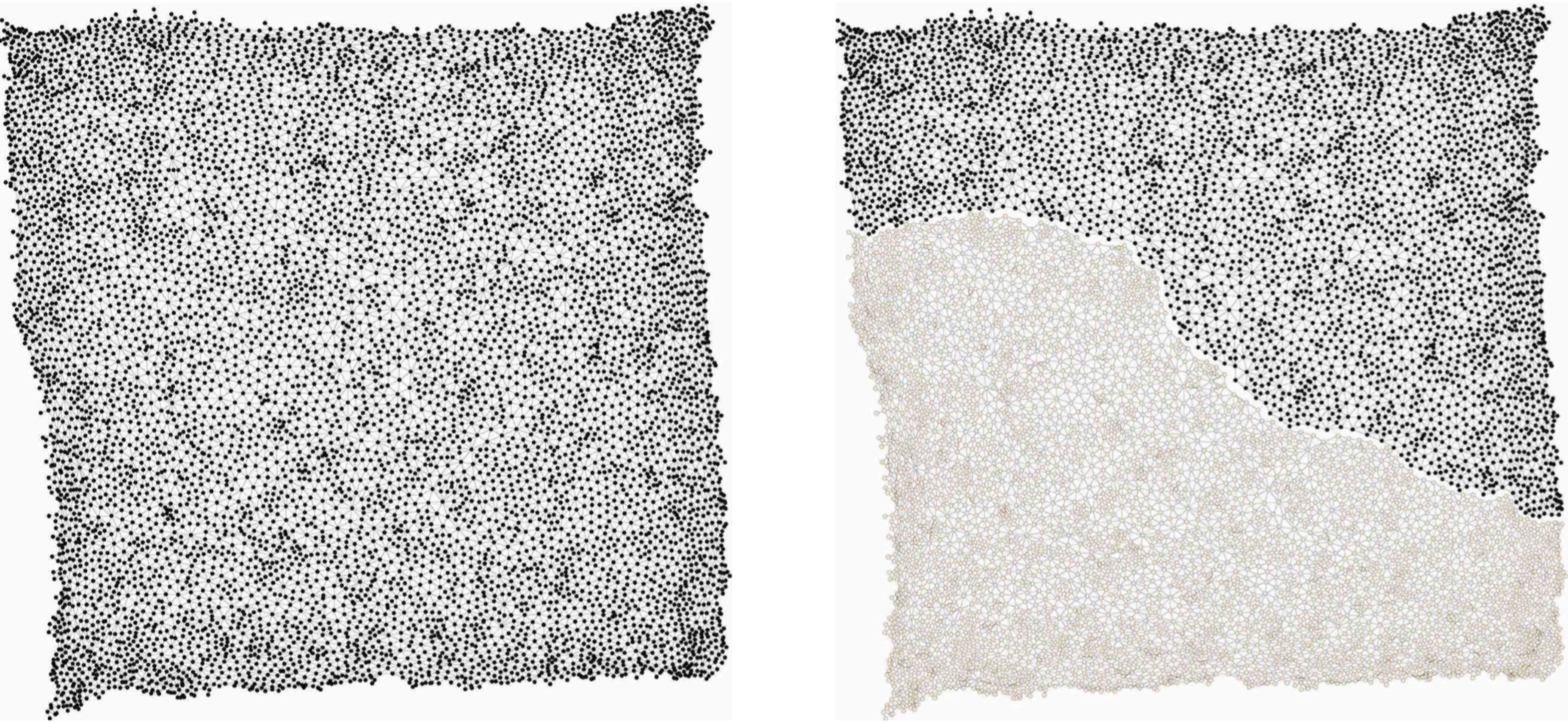
$$\approx$$

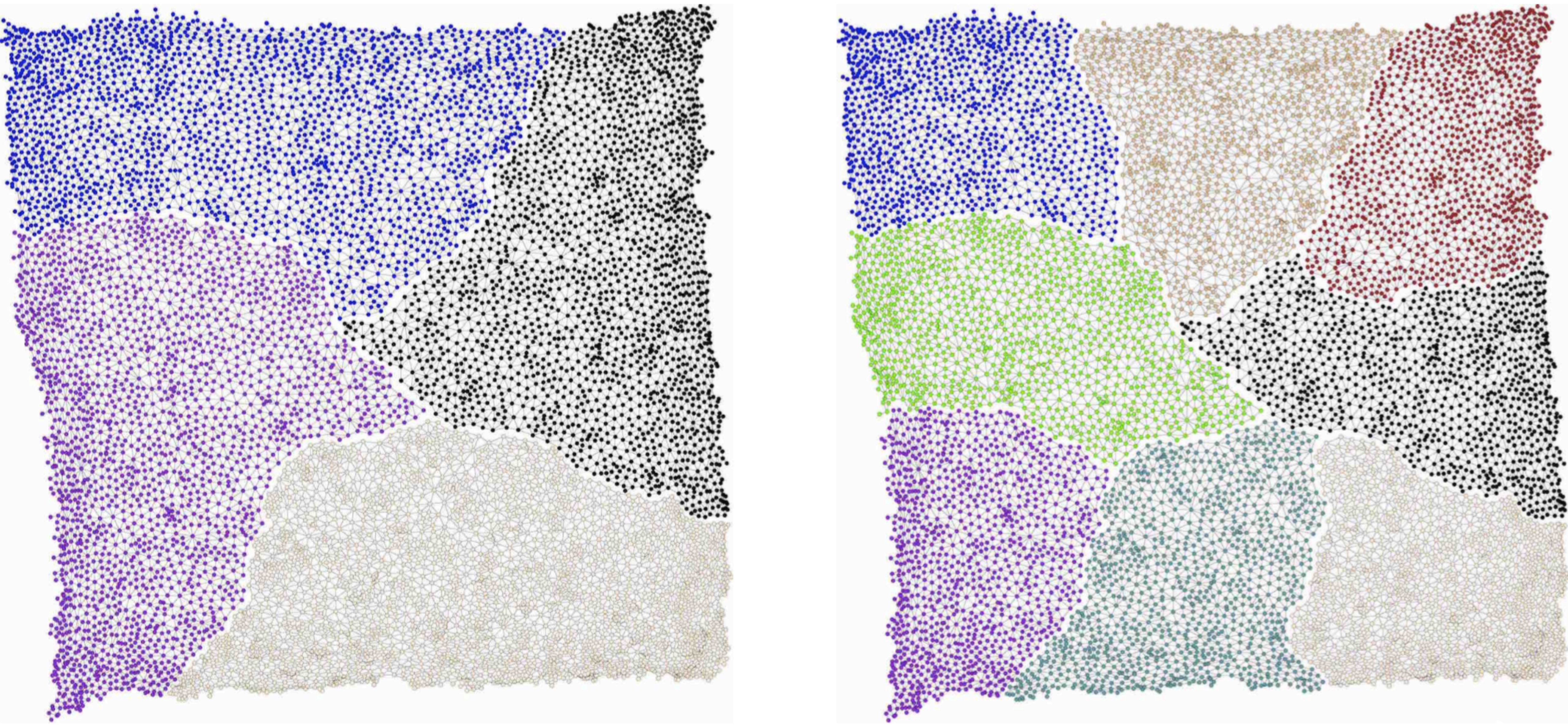


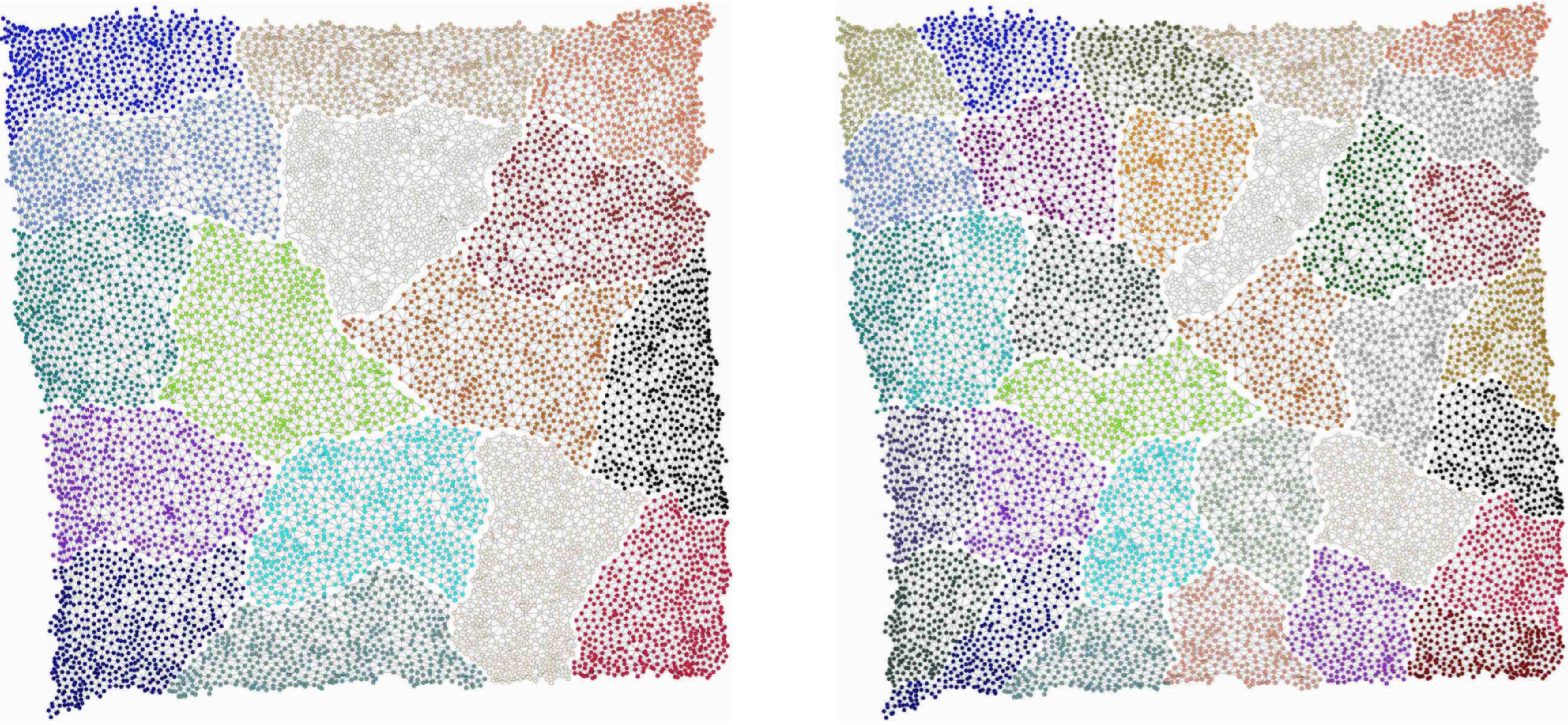
$$\begin{bmatrix} A_{22} & & U & & & \\ & A_{33} & & V & & \\ U^\top & & & & -I & \\ & V^\top & & & & -I \\ & & -I & & & K \\ & & & -I & K^\top & \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ z_2 \\ z_3 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hierarchical Extended-Sparsification

Practical Algorithm

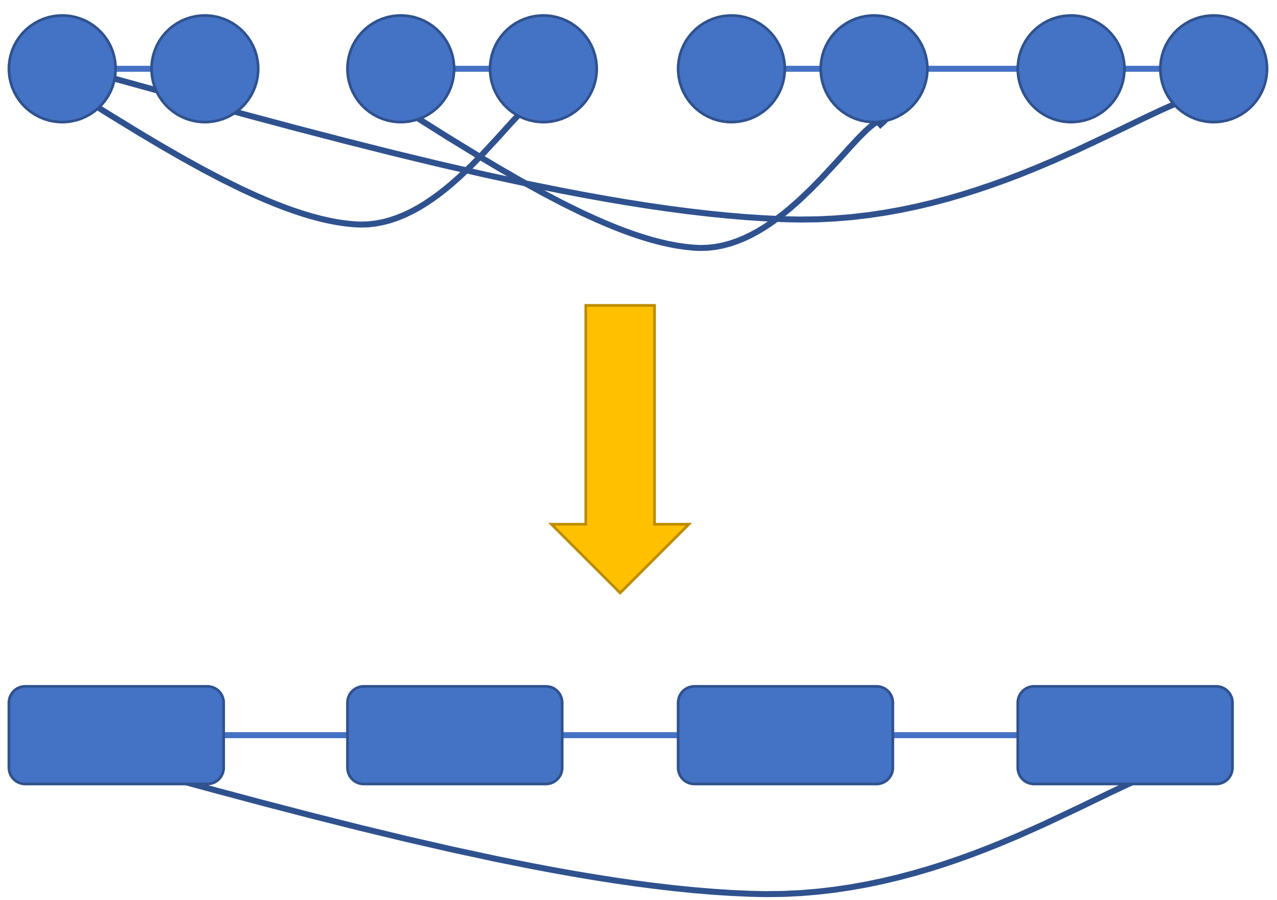






Merge

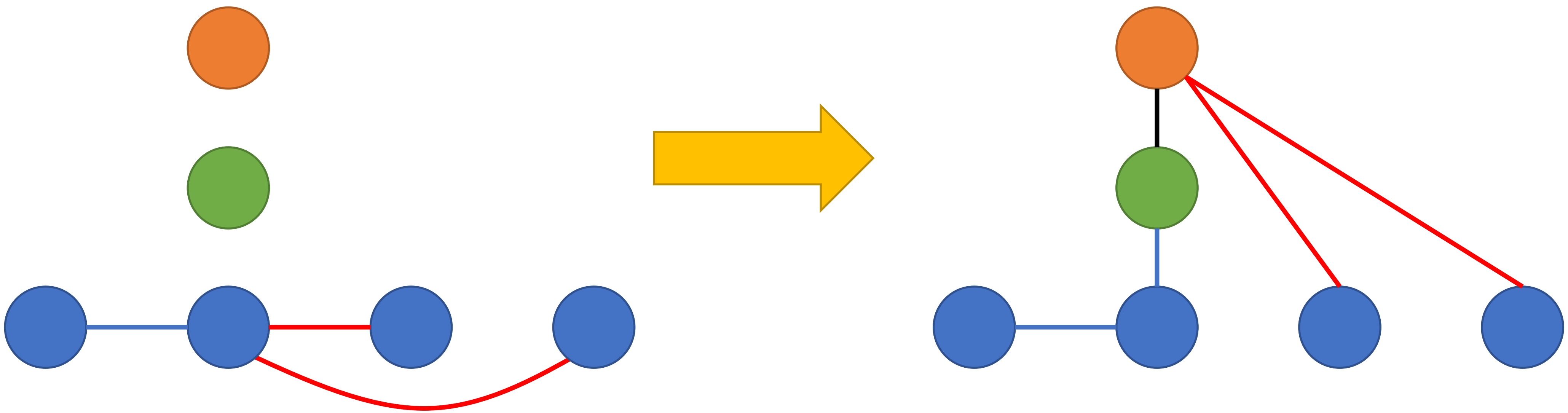
Merge nodes following bissection tree



Compress

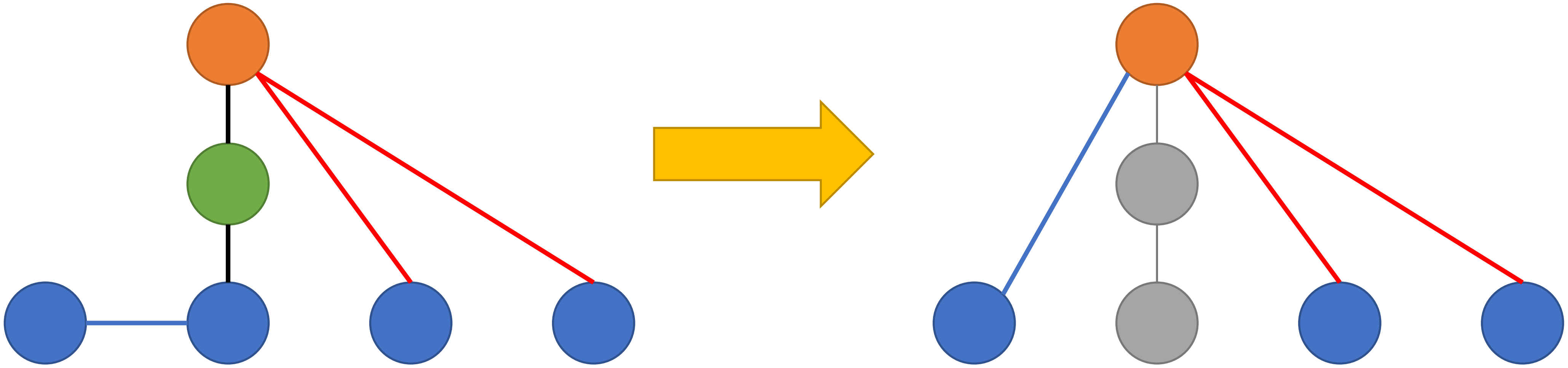
Low-Rank approximation for far-field interactions

$$D \approx UK^{\top}$$

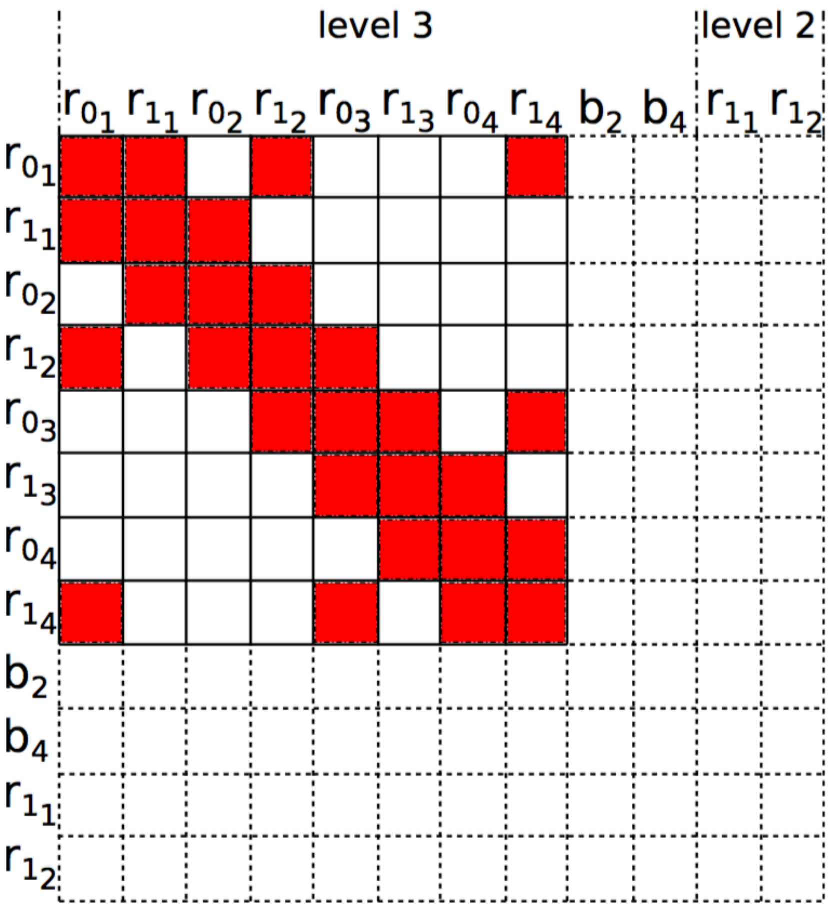


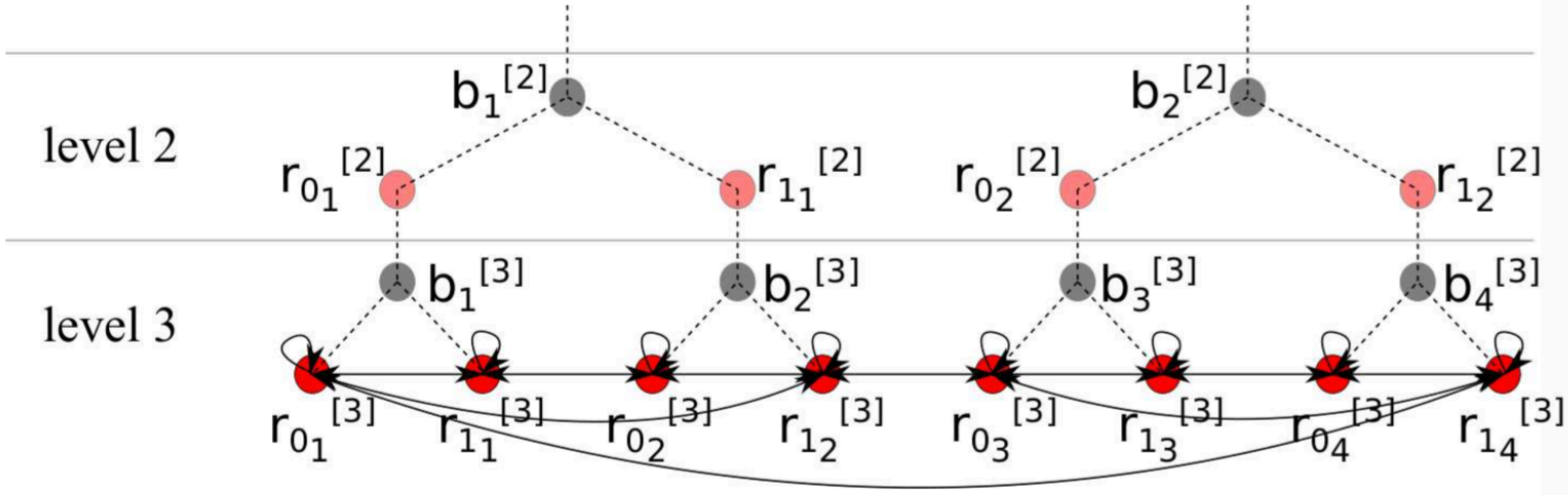
Eliminate

Usual LU/Cholesky elimination



Merge-Compress-Eliminate





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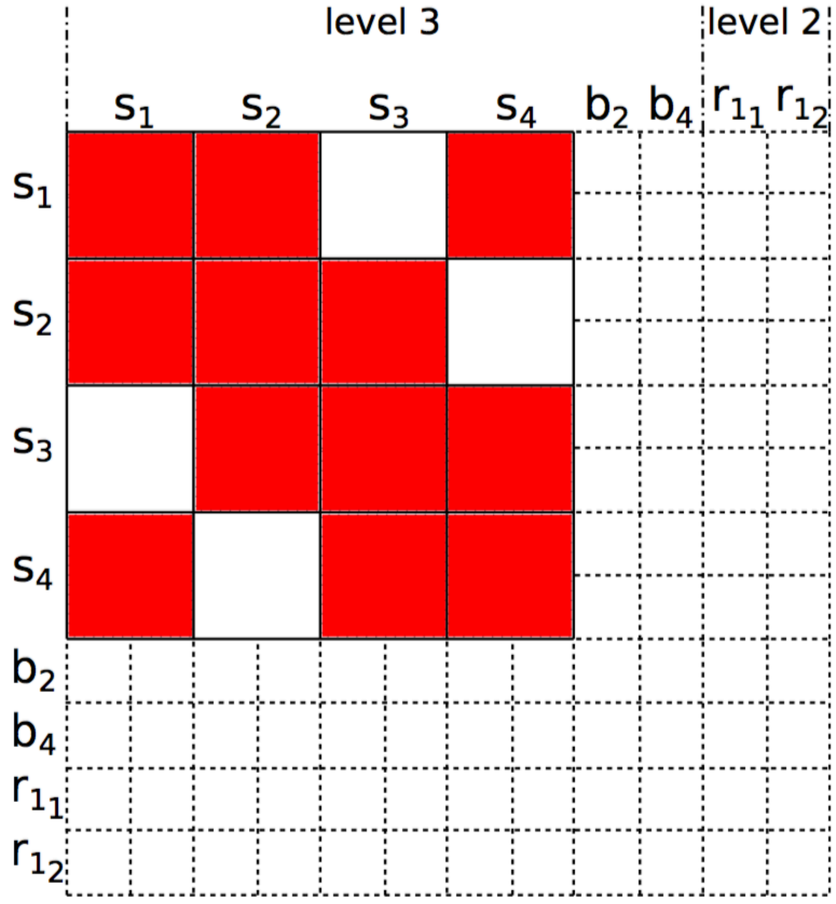

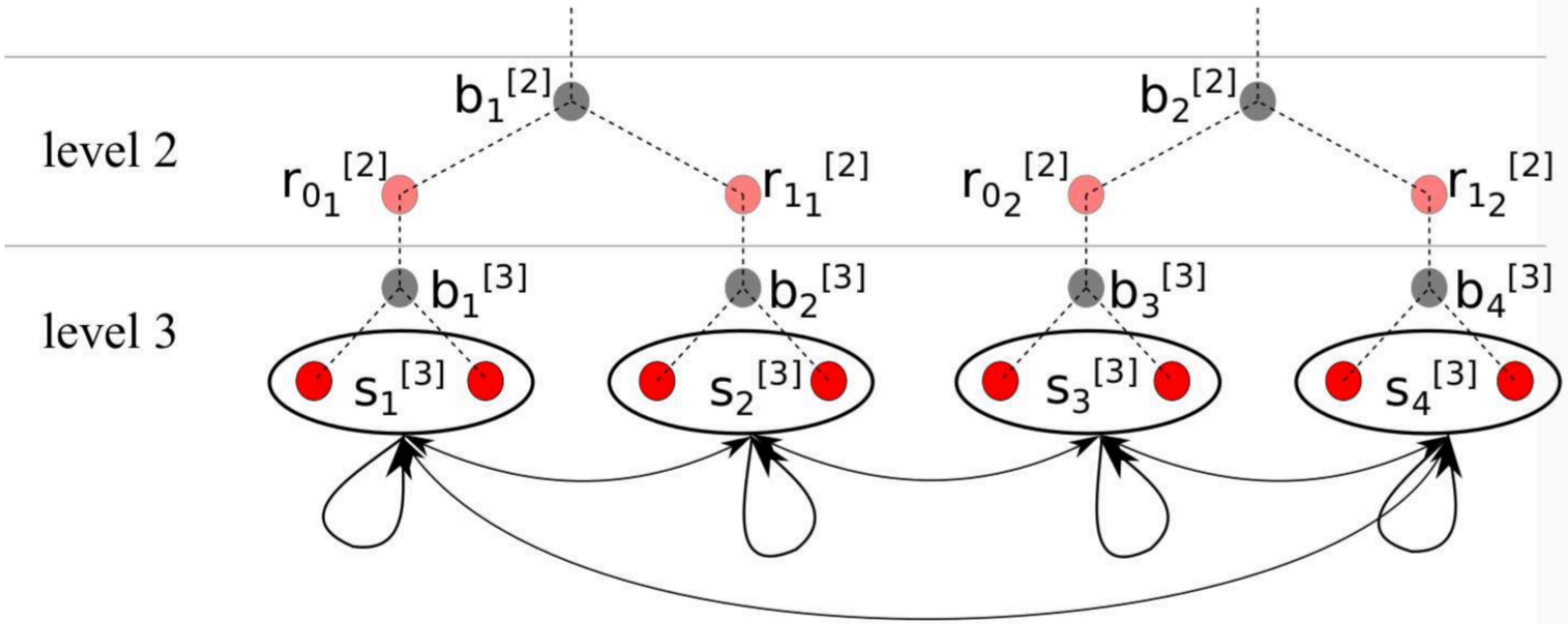

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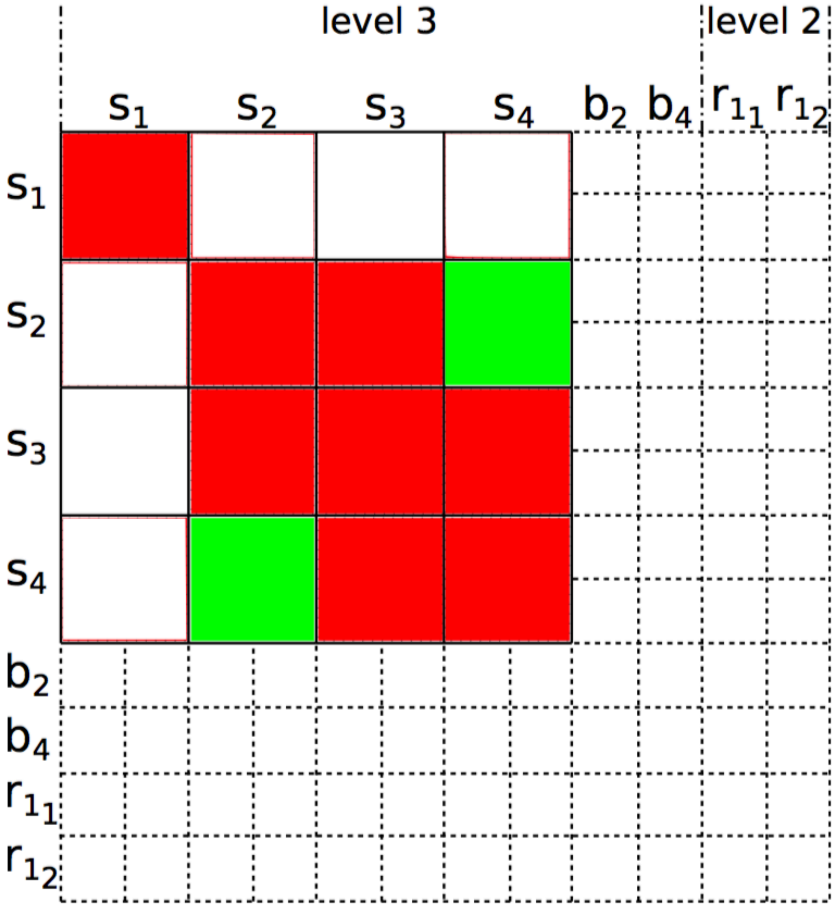

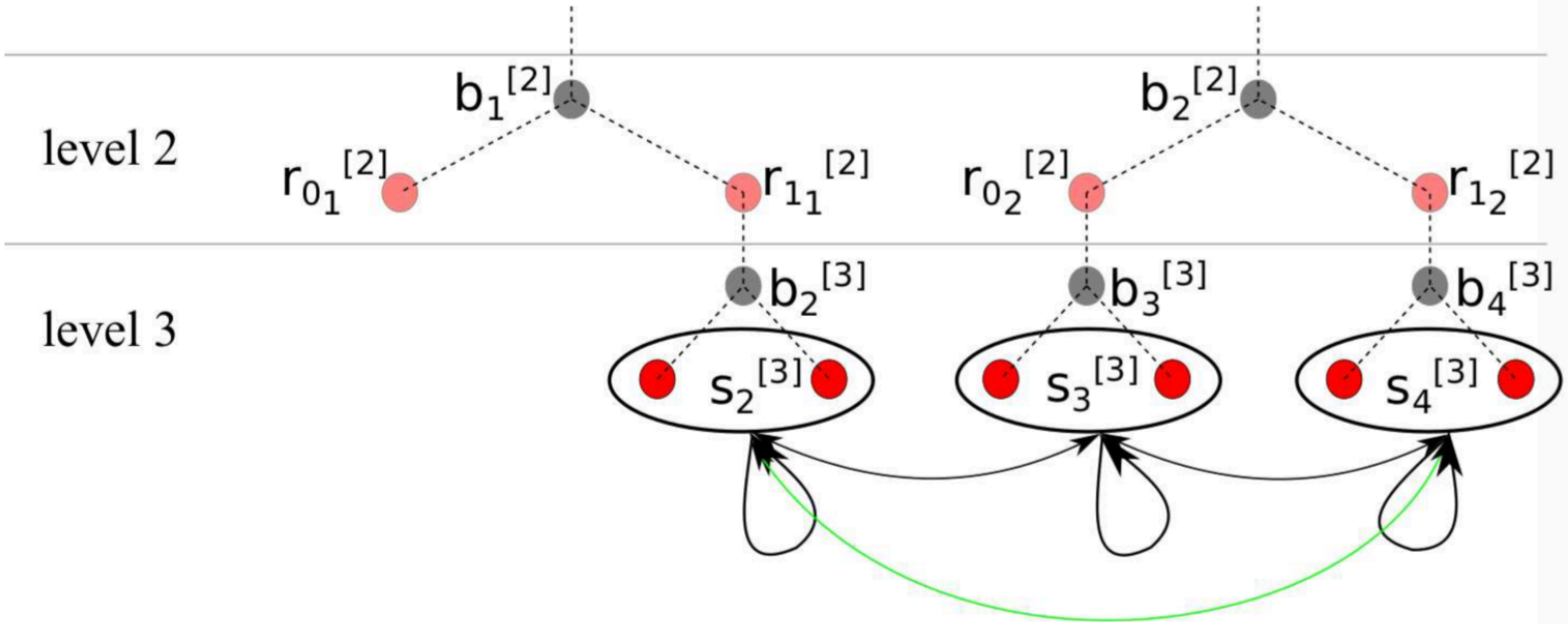

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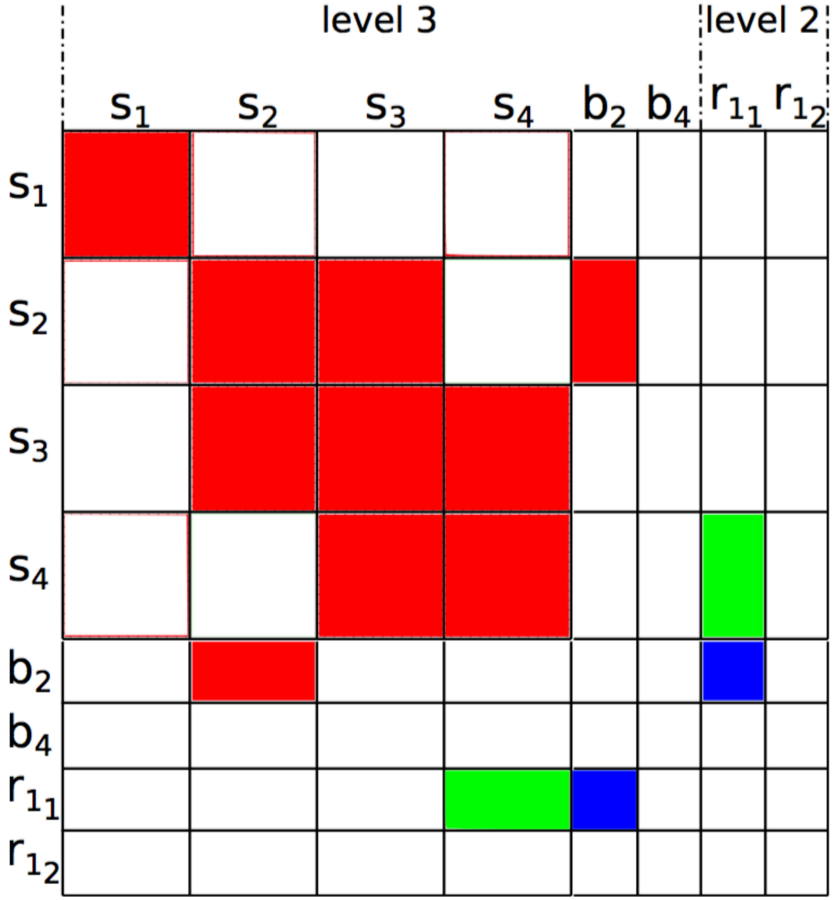

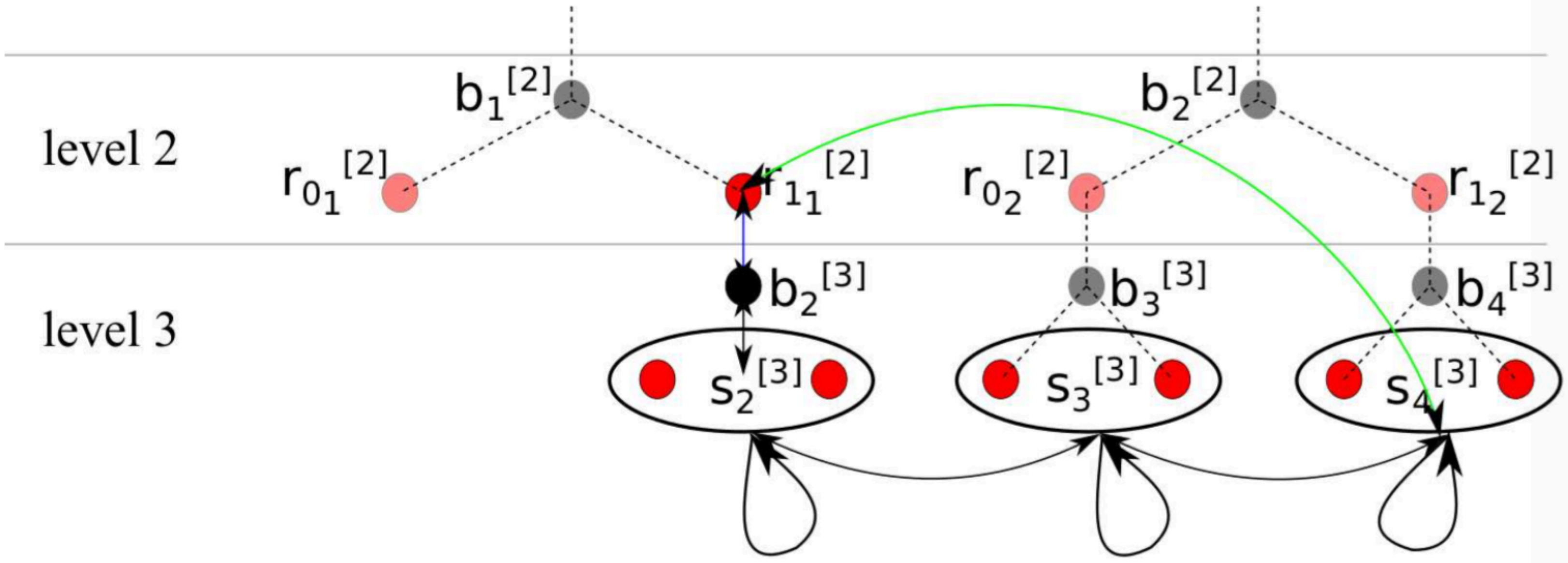

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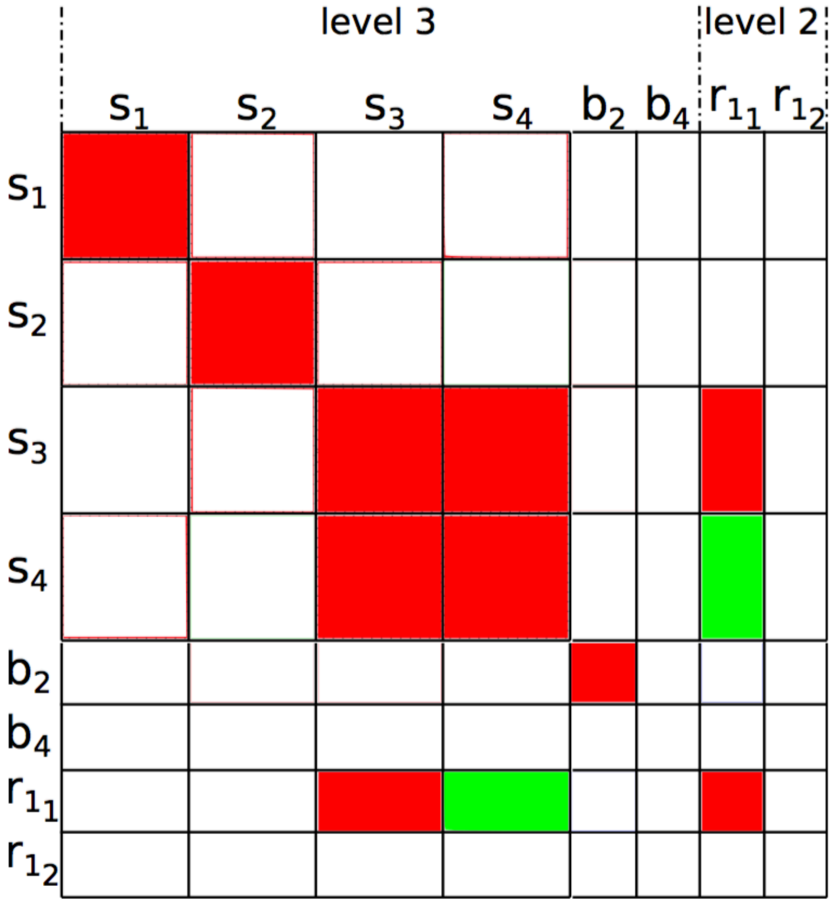

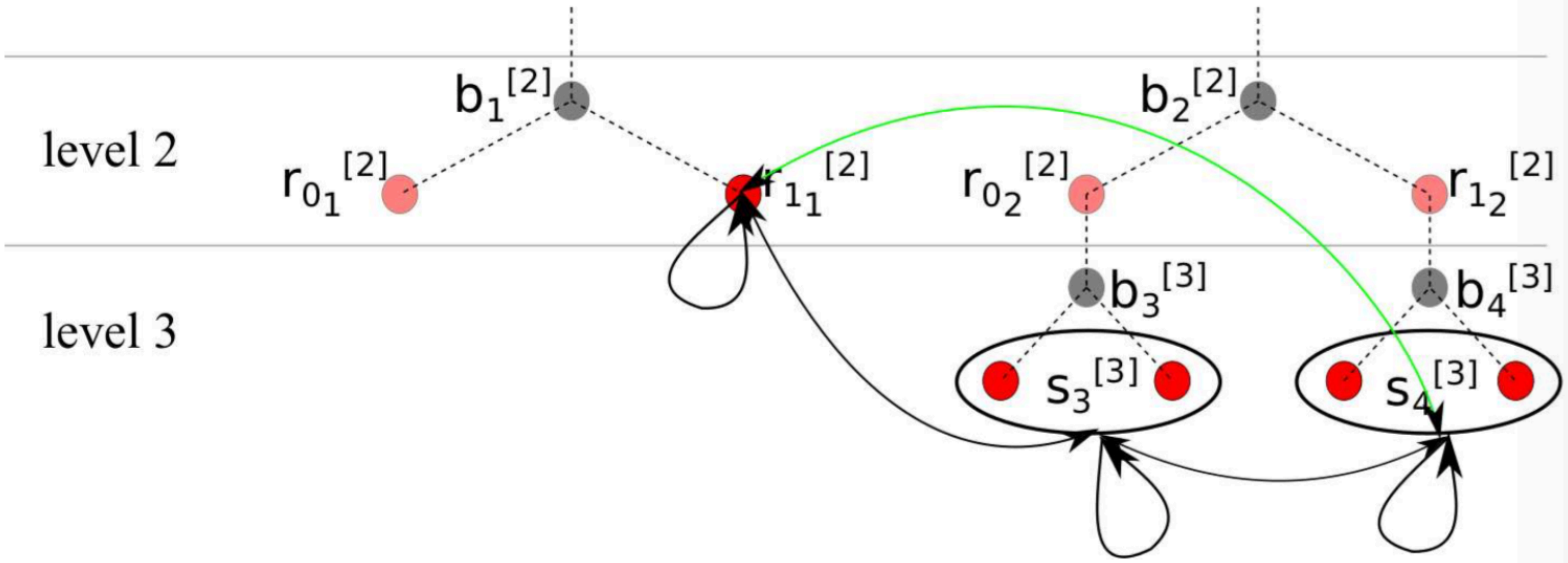

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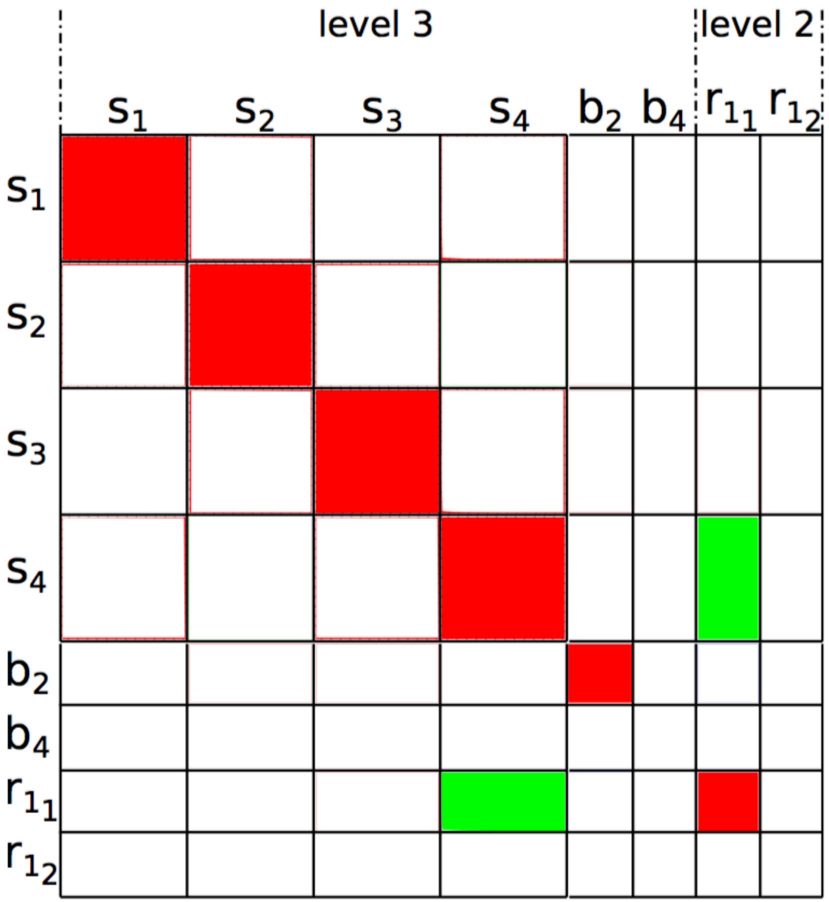

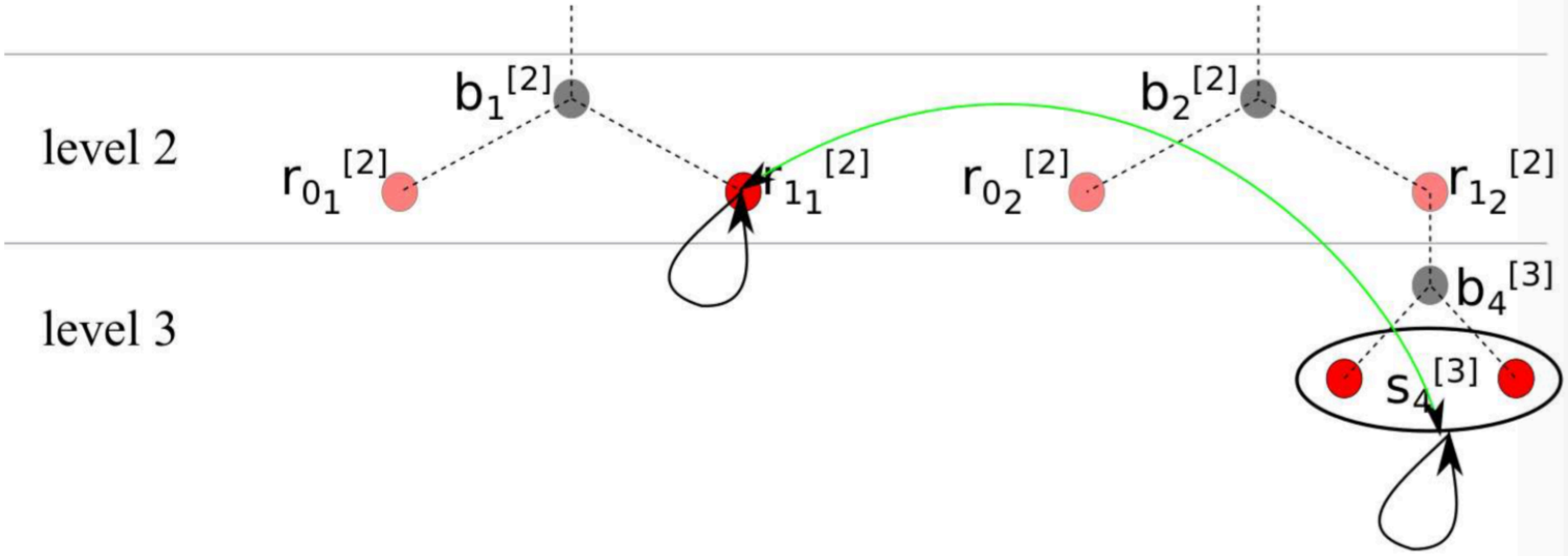

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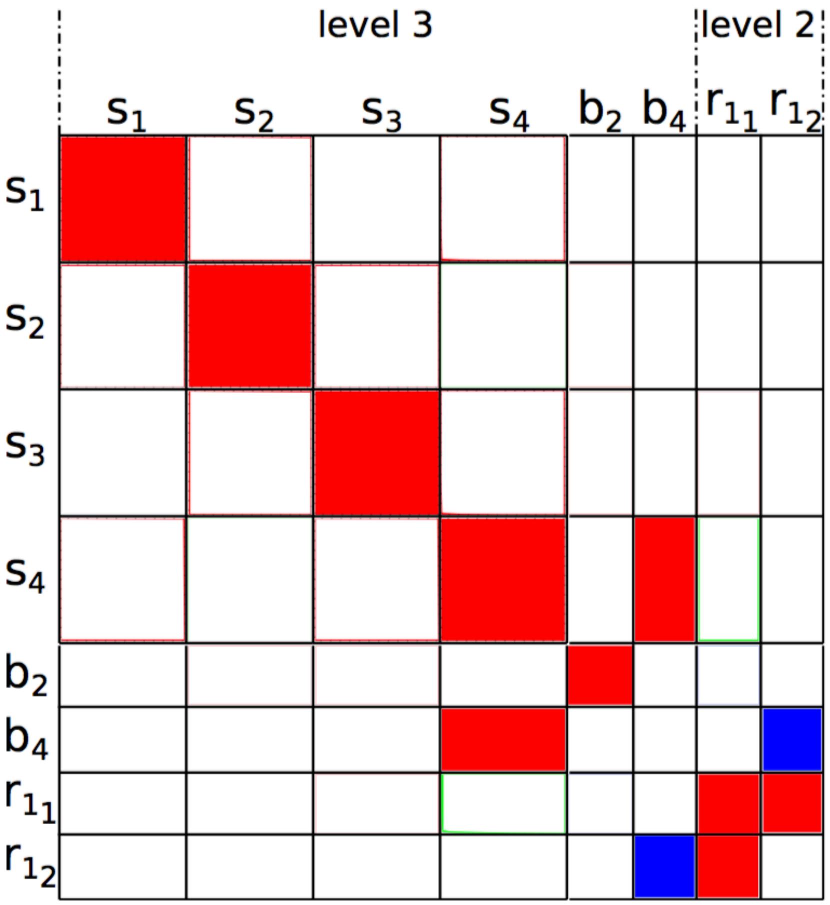

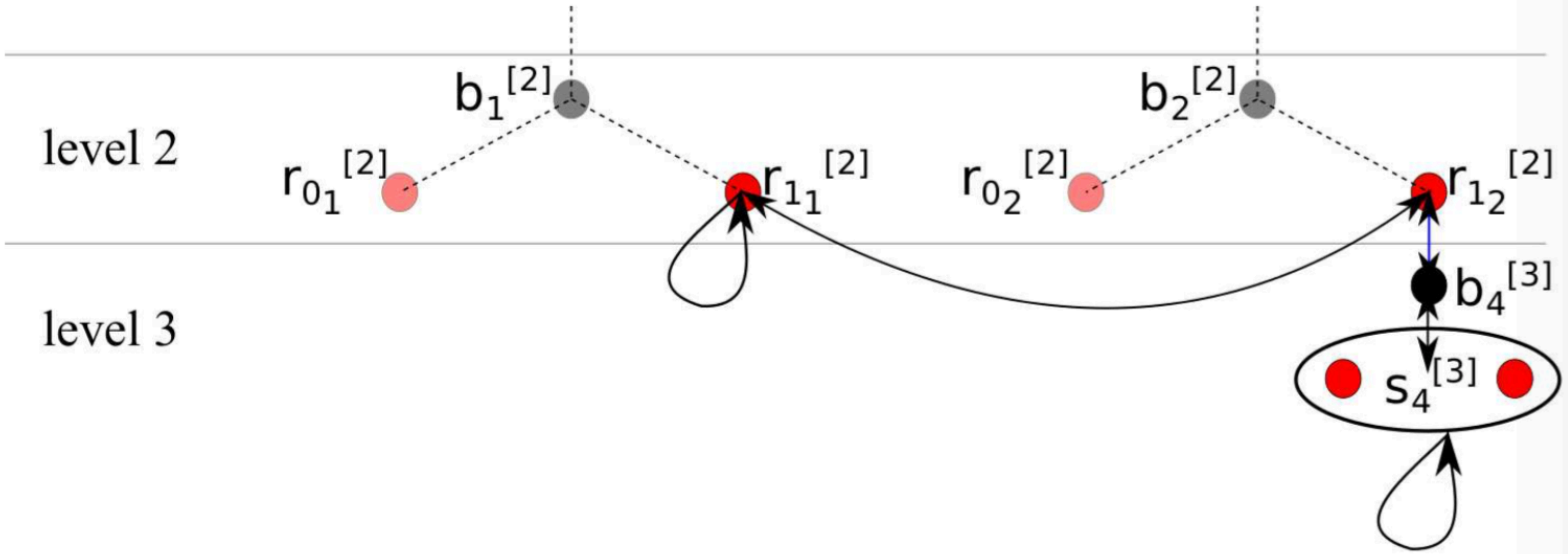

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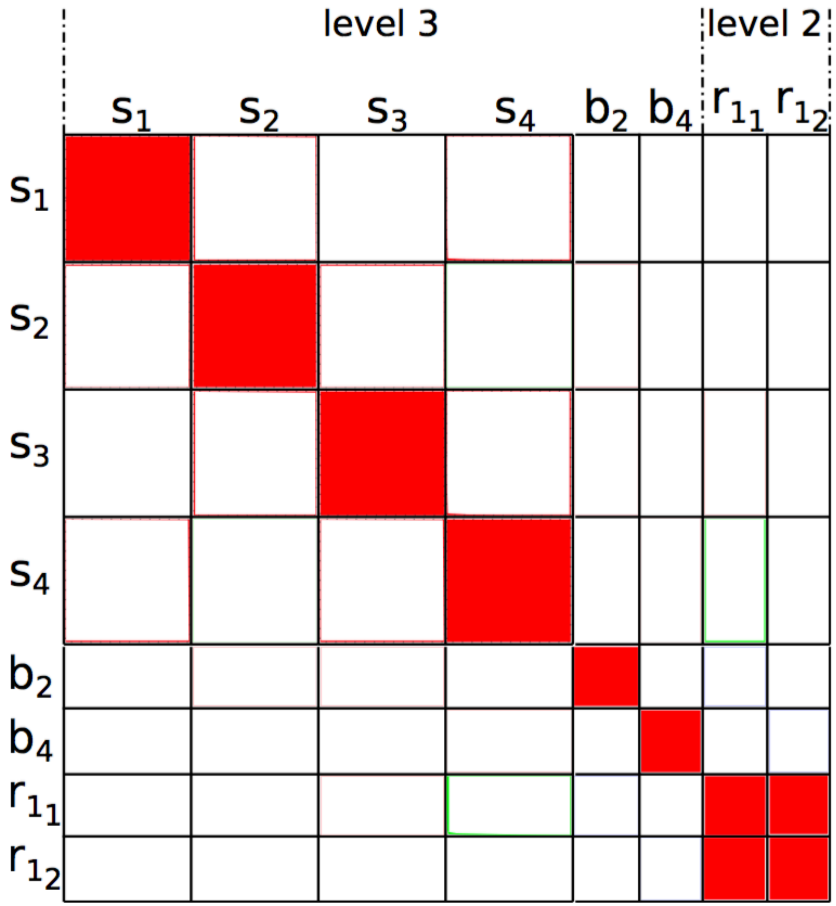
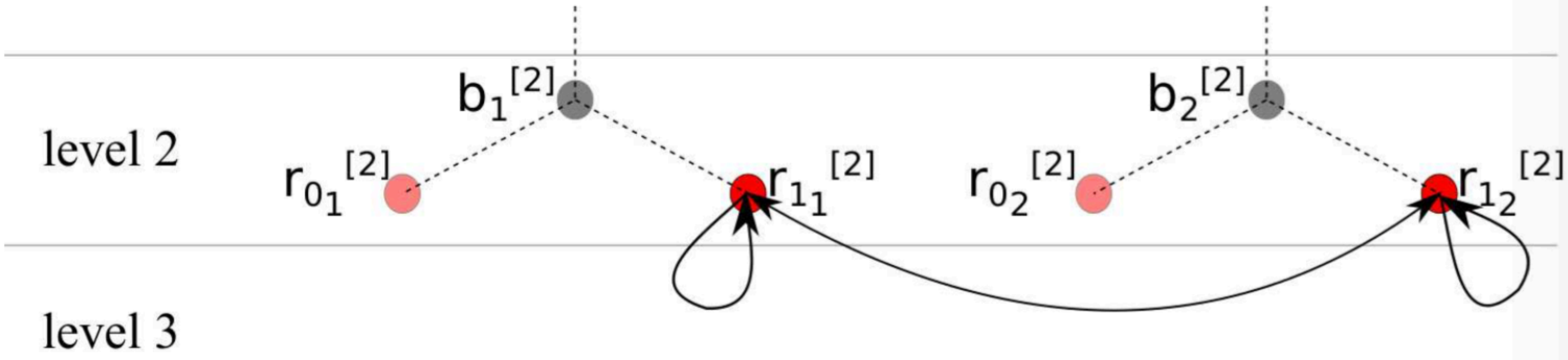




Application to Extruded Meshes & Ice-Sheet Modeling

A (very) ill-conditionned challenging problem

Ice-sheet modeling

- Incompressible, low-Reynold numbers viscous flow

$$\begin{cases} -\nabla \cdot (2\mu \dot{\epsilon_1}) + \rho g \dfrac{\partial s}{\partial x} = 0 \\ -\nabla \cdot (2\mu \dot{\epsilon_2}) + \rho g \dfrac{\partial s}{\partial y} = 0 \end{cases}$$

$$\dot{\epsilon_1} = \begin{pmatrix} 2\dot{\epsilon_{xx}} + \dot{\epsilon_{yy}} \\ \dot{\epsilon_{xy}} \\ \dot{\epsilon_{xz}} \end{pmatrix}$$

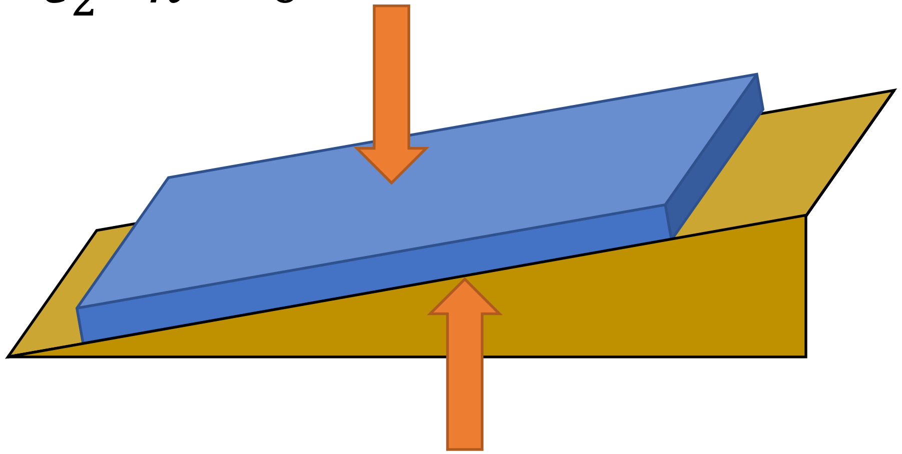
$$\dot{\epsilon_2} = \begin{pmatrix} \dot{\epsilon_{xy}} \\ \dot{\epsilon_{xx}} + 2\dot{\epsilon_{yy}} \\ \dot{\epsilon_{xz}} \end{pmatrix}$$

$$\dot{\epsilon}_{xx} = \frac{\partial u}{\partial x}, \dot{\epsilon}_{yy} = \frac{\partial v}{\partial y}, \dot{\epsilon}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \dot{\epsilon}_{xz} = \frac{1}{2}\frac{\partial u}{\partial z}, \dot{\epsilon}_{yz} = \frac{1}{2}\frac{\partial v}{\partial z}$$

Ice-sheet modeling

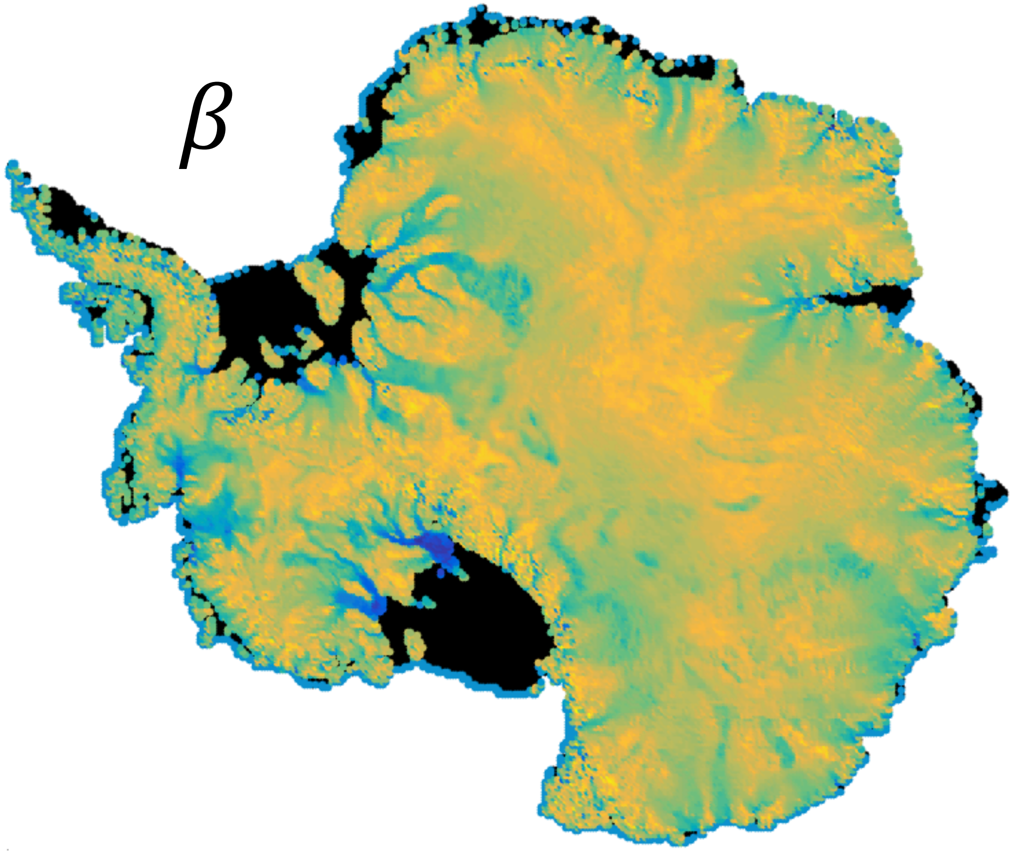
$$\dot{\epsilon_1} \cdot n = 0$$
$$\dot{\epsilon_2} \cdot n = 0$$

Free surface (top)

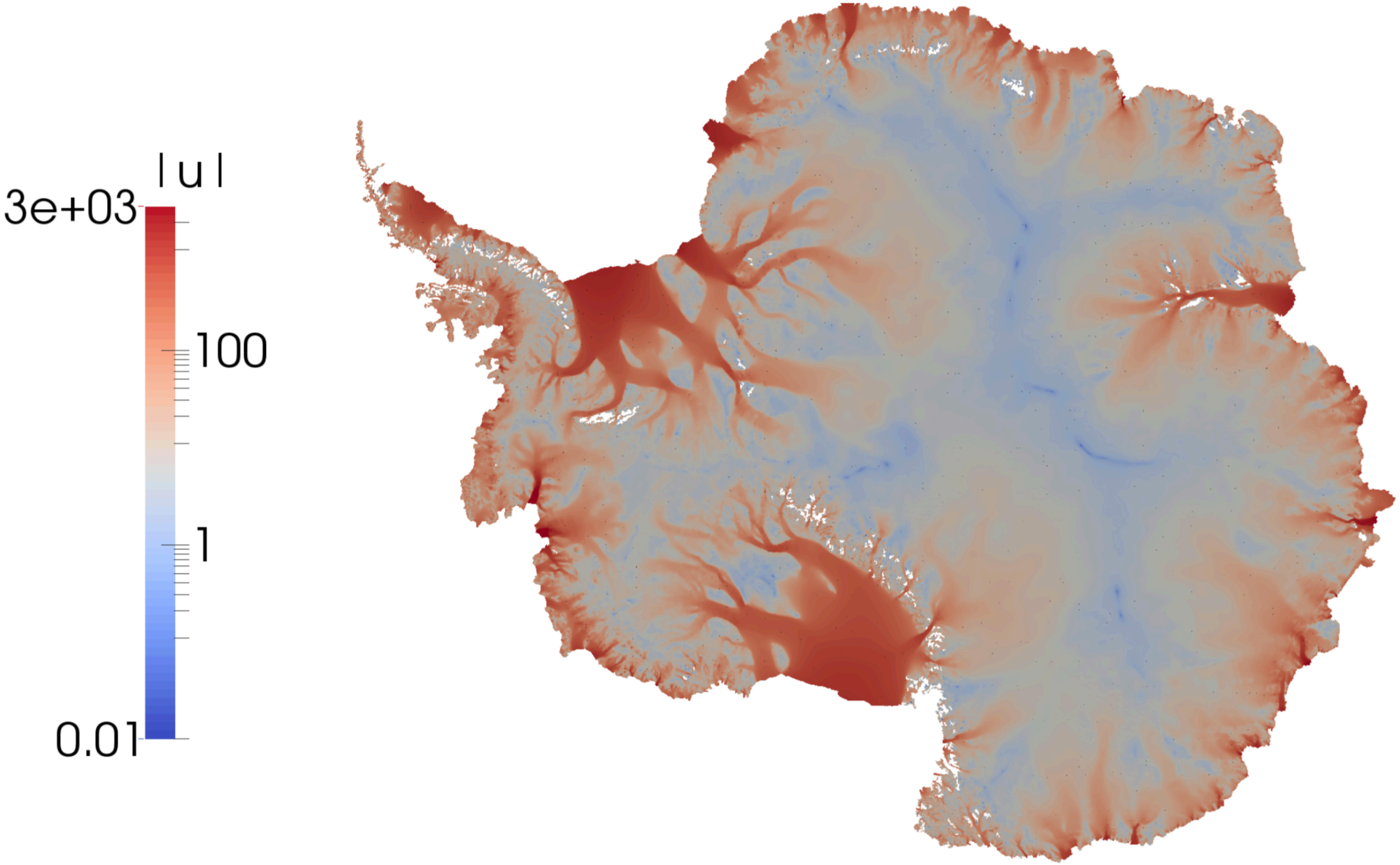


$$2\mu\dot{\epsilon_1} \cdot n + \beta u = 0$$
$$2\mu\dot{\epsilon_2} \cdot n + \beta u = 0$$

Friction (bottom)



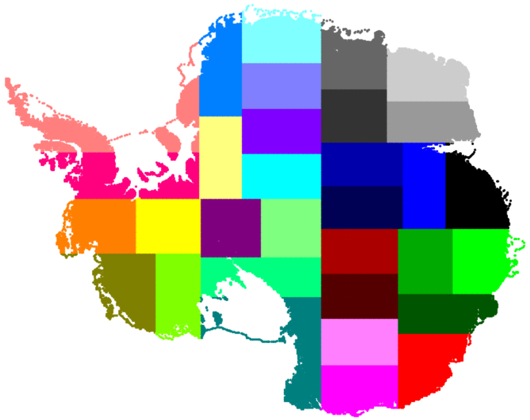
Ice-sheet modeling: solution



Vanilla algorithm

Resolution	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
64 km	320	102	13	6

Vertical clustering



Resolution	Block ILU	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
64 km	12	12	12	11	11
32 km	25	26	22	21	17
16 km	50	52	44	37	28
8 km	107	107	83	71	35

Diagonal scaling

We have a choice to make

No Scaling

$$\begin{bmatrix} A_{22} & D^T \\ D & A_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

$$D \approx UKV^T$$

Diagonal Scaling

$$\begin{bmatrix} A_{22} & D^T \\ D & A_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

$$A_{22} = LL^T$$
$$DL^{-T} \approx UKV^T$$

Diagonal scaling

Resolution	Block ILU	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
64 km	12	14	10	9	5
32 km	25	21	12	8	5
16 km	50	37	14	7	5
8 km	107	54	16	8	6
4 km	190	99	22		

Diagonal scaling (solve time)

Resolution	Block ILU	$\varepsilon = 10^{-2}$
64 km	0.7	1.3
32 km	6.1	7
16 km	51	44
8 km	562	268

Near-nullspace preservation

Nullspace without BC's

$$\Delta \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A\phi \approx 0$$

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$\phi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -y_i \\ x_i \end{bmatrix}$$

Near-nullspace preservation

$$\begin{bmatrix} A_{xx} & A_{xy}^T \\ A_{xy} & A_{yy} \end{bmatrix}$$

$$A_{xy} \approx B$$

$$B\phi_y = A_{xy}\phi_y \qquad B^\top\phi_x = A_{xy}^\top\phi_x$$

Near-nullspace preservation: how?

- QR Factorization to have exact range/nullspace

$$U_1 R = [\phi_x \quad A_{xy}\phi_y] \qquad\qquad K_2^\top = U_1^\top A_{xy}$$

- Do SVD (or other) to approximate the rest

$$U_2 K_2^\top \approx (I - U_1 U_1^\top) A_{xy}$$

- Low-Rank approx is

$$A_{xy} \approx [U_1 \quad U_2] \begin{bmatrix} K_1^\top \\ K_2^\top \end{bmatrix} = UK^\top \quad \Rightarrow \quad \begin{aligned} A_{xy}\phi_y &= UK^\top \phi_y \\ A_{yx}\phi_x &= KU^\top \phi_x \end{aligned}$$

Near-nullspace preservation

Nullspace + Scaling

Resolution	Block ILU	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
64 km	12	11	12	8
32 km	25	13	11	8
16 km	51	18	11	8

References

- H. Pouransari, P. Coulier, And E. Darve. “Fast Hierarchical Solvers For Sparse Matrices Using Extended Sparsification And Low-rank Approximation”, Siam J. Sci. Comput, Vol. 39, No. 3, pp. A797–A830
- R. Tuminaro, M. Perego, I. Tezaur, A. Salinger, And S. Price. “A Matrix Dependent/Algebraic Multigrid Approach For Extruded Meshes With Applications To Ice Sheet Modeling”. Siam J. Sci. Comput
- K. Yang, H. Pouransari, and E. Darve. "Sparse hierarchical solvers with guaranteed convergence." *arXiv preprint arXiv:1611.03189* (2016).
- Paper in preparation with presented results