An Algebraic Sparsified Nested Dissection Algorithm using Low-Rank Approximations

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Linear Systems

We want to solve $Ax = b$

Direct Methods | Approximate Factorizations | Iterative methods

Direct Methods

(Sparse) LU + Ordering

- Robust
- Accurate
- Generic
- Costly

Approximate Factorizations

ILU / ND + H-algebra …

- Tunable accuracy
- Tunable cost
- Relatively generic

Iterative methods

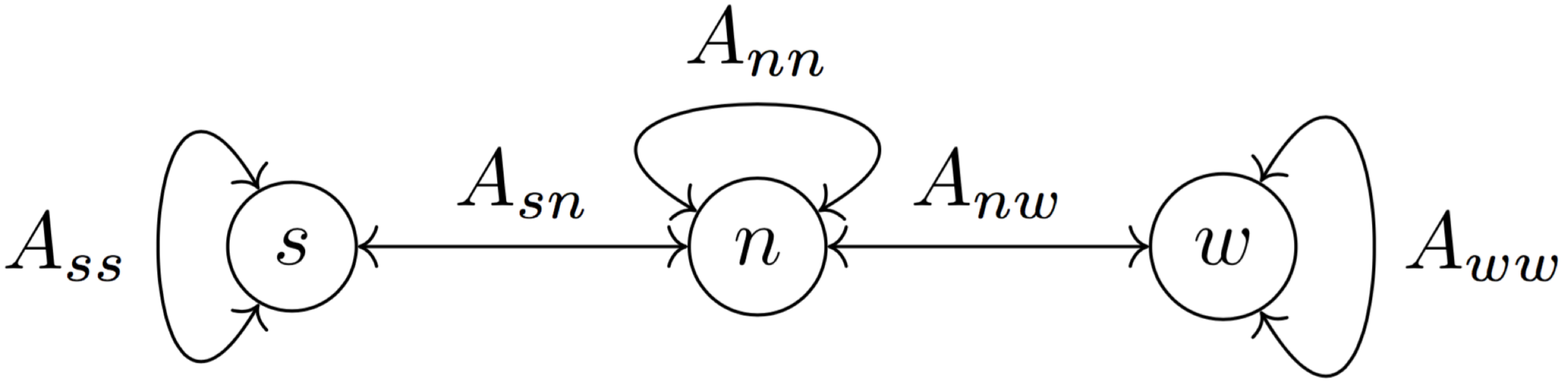
+ Custom Preconditioner

- Cheap
- Specific

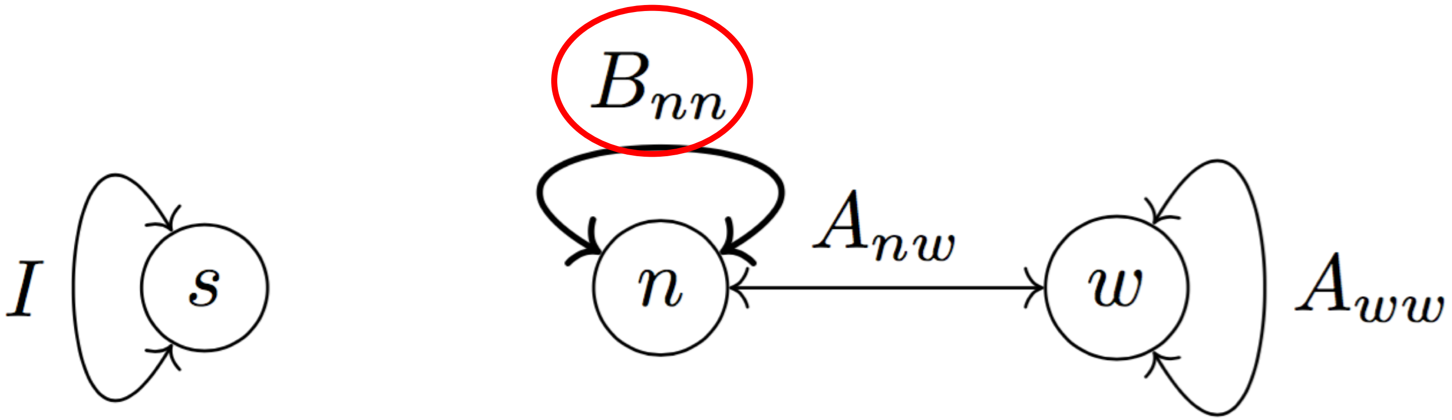
Our approach is heavily inspired by K. L. Ho and L. Ying, Hierarchical interpolative factorization for elliptic operators: differential equations, Communications on Pure and Applied Mathematics, 69 (2016), pp. 1415–1451

Sparse Linear Systems

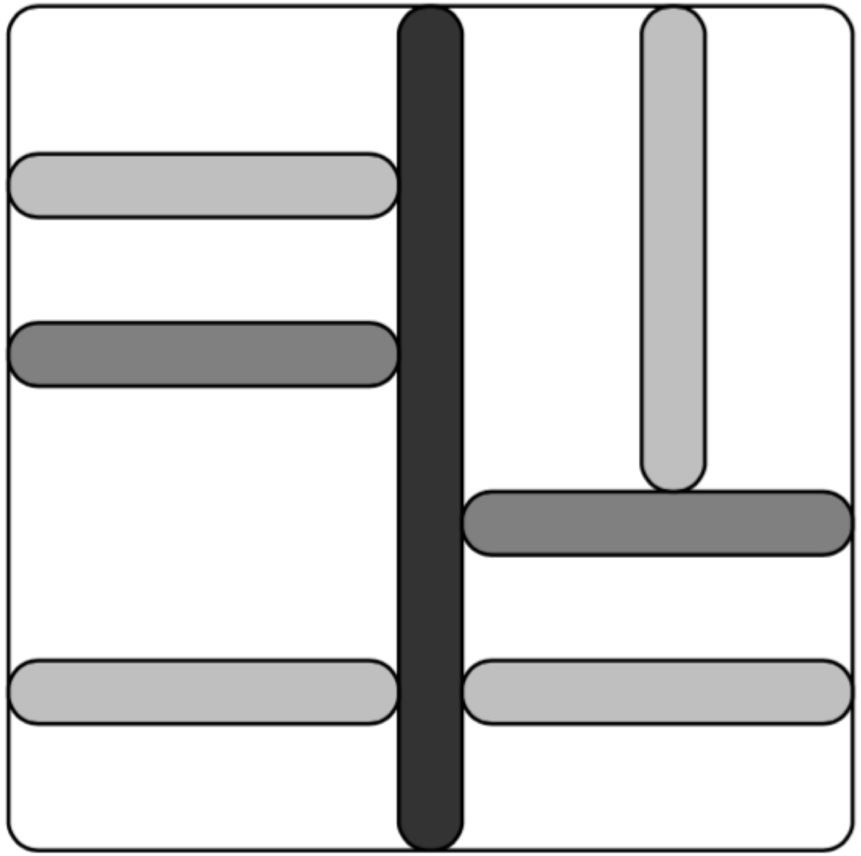
$$\begin{bmatrix} A_{ss} & & A_{sn} \\ & A_{ww} & A_{wn} \\ A_{ns} & A_{nw} & A_{nn} \end{bmatrix}$$

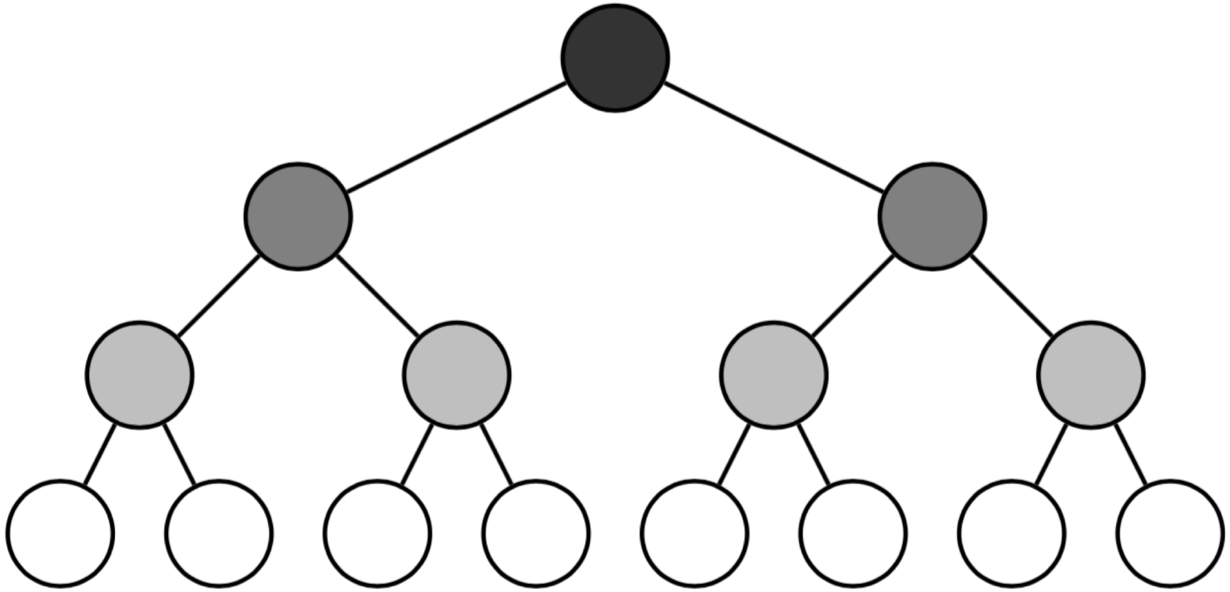


$$\begin{bmatrix} I & & \\ & A_{ww} & A_{wn} \\ & A_{nw} & B_{nn} \end{bmatrix}$$



Nested Dissection

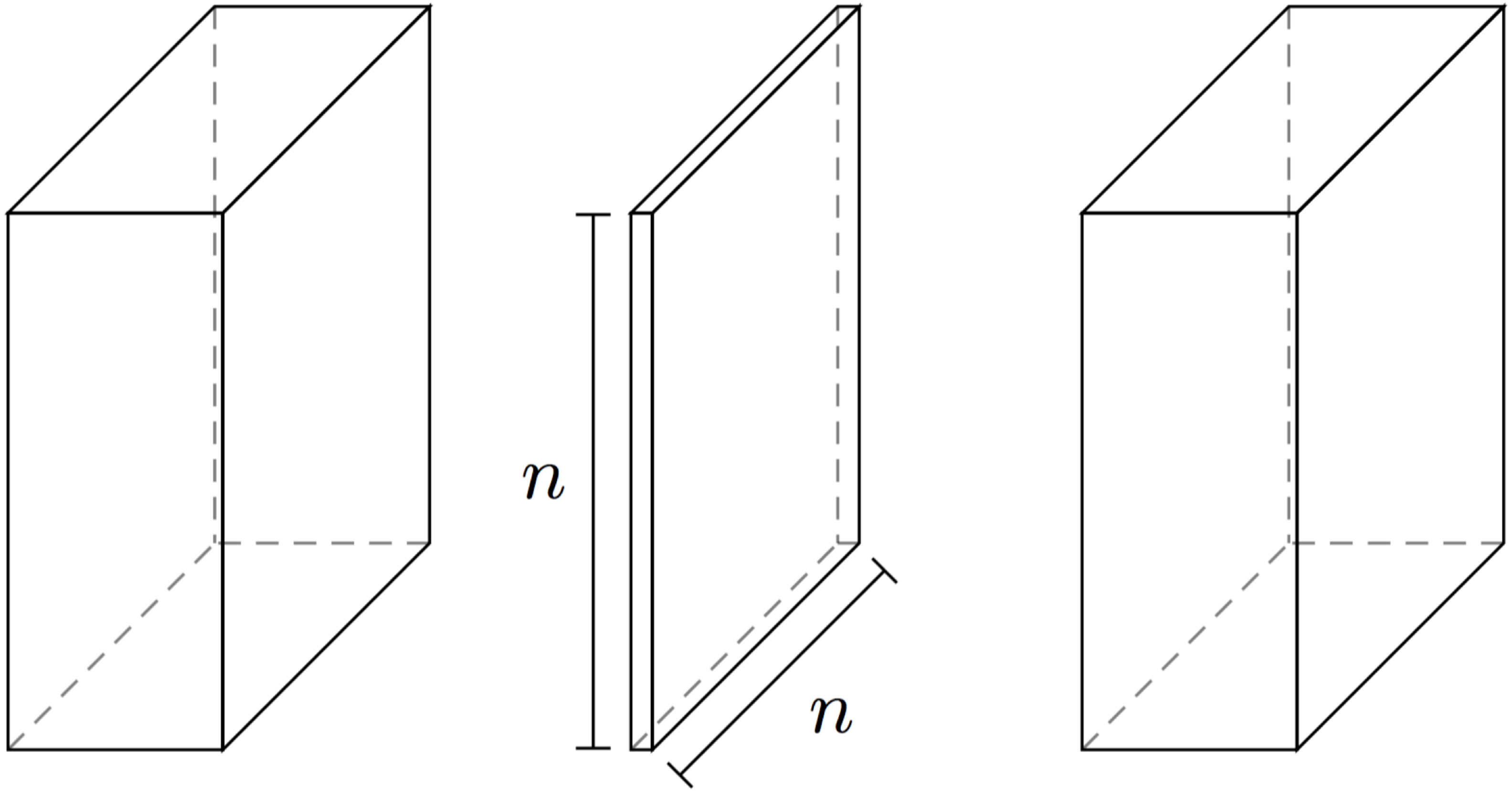




Nested Dissection

Issue: separators are small, but still too big on typical 3D problems

$N = n^3$
Separator: n^2
Fact. cost: $n^{2 \cdot 3} = N^2$



Sparsification I

(1) We start with

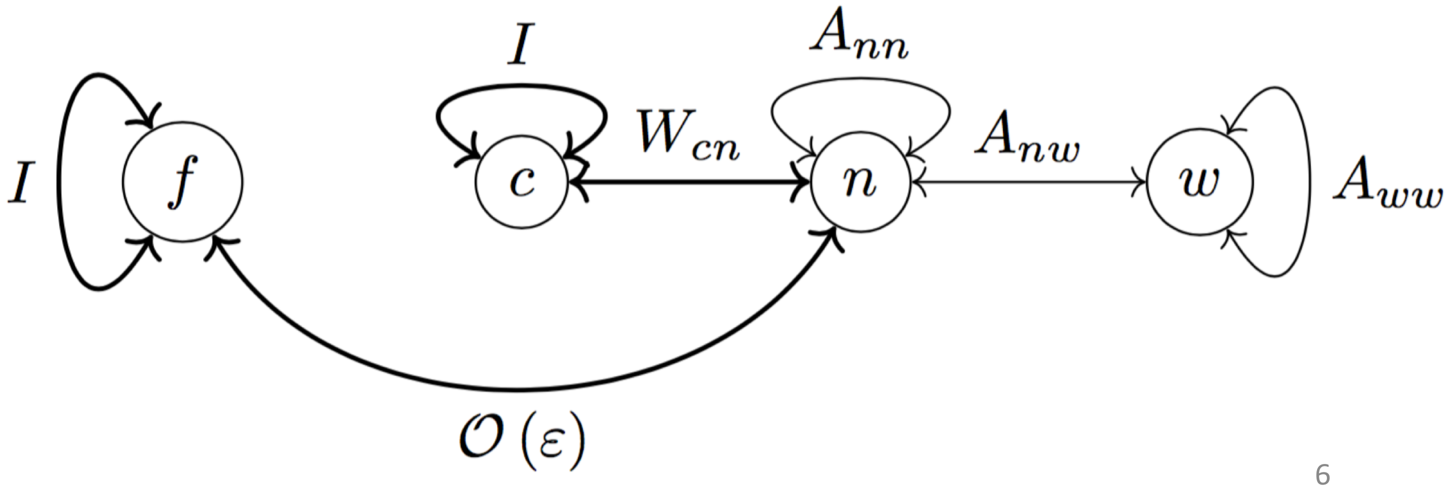
$$\begin{bmatrix} I & & A_{sn} \\ & A_{ww} & A_{wn} \\ A_{ns} & A_{nw} & A_{nn} \end{bmatrix}$$

(2) We then approximate

$$A_{sn} = Q_{sc} W_{cn} + \varepsilon$$
$$Q^{\top} s = f \cup c$$

(3) We end up with

$$\begin{bmatrix} I & & & \varepsilon \\ & I & & W_{cn} \\ & & A_{ww} & A_{wn} \\ \varepsilon & W_{cn}^{\top} & A_{nw} & A_{nn} \end{bmatrix}$$



Sparsification I

$$\underbrace{\begin{bmatrix} L_{ss}^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} A_{ss} & & A_{sn} \\ & A_{ww} & A_{wn} \\ A_{ns} & A_{nw} & A_{nn} \end{bmatrix} \begin{bmatrix} L_{ss}^{-\top} & & \\ & I & \\ & & I \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} Q^{\top} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} I & & A_{sn} \\ & A_{ww} & A_{wn} \\ A_{ns} & A_{nw} & A_{nn} \end{bmatrix} \begin{bmatrix} Q & & \\ & I & \\ & & I \end{bmatrix}}$$

$$\begin{bmatrix} I & & & \varepsilon \\ & I & & W_{cn} \\ & & A_{ww} & A_{wn} \\ \varepsilon & W_{cn}^{\top} & A_{nw} & A_{nn} \end{bmatrix}$$

Sparsification II

(1) We start with

$$\begin{bmatrix} A_{ss} & & A_{sn} \\ & A_{ww} & A_{wn} \\ A_{ns} & A_{nw} & A_{nn} \end{bmatrix}$$

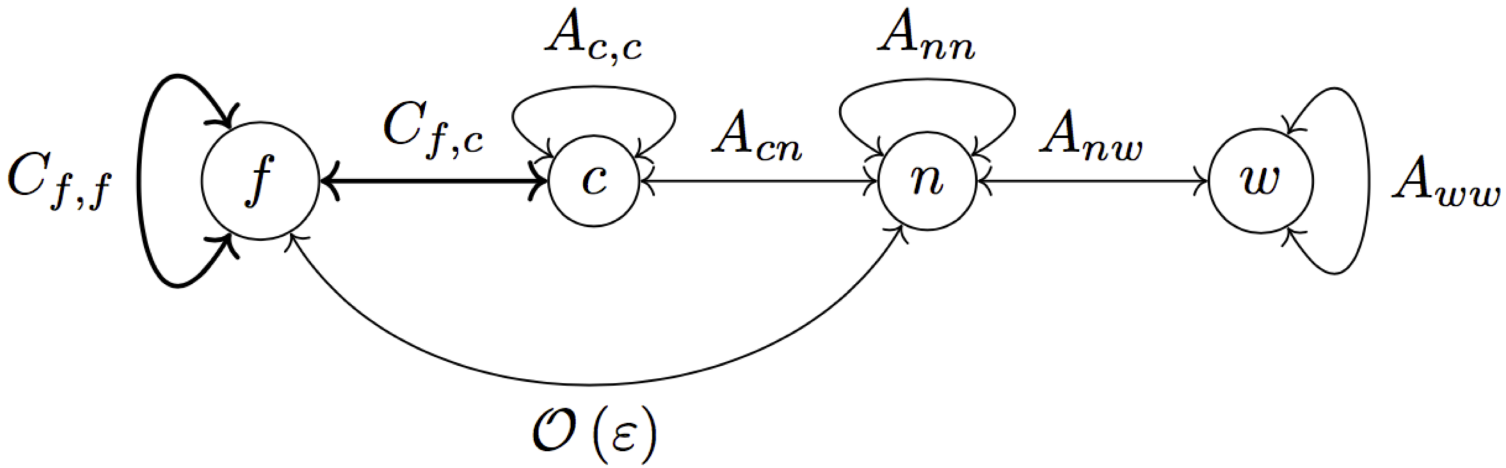
(2) We then approximate

$$A_{sn} = \begin{pmatrix} T_{fc} \\ I \end{pmatrix} A_{cn} + \varepsilon$$

$$s = f \cup c$$

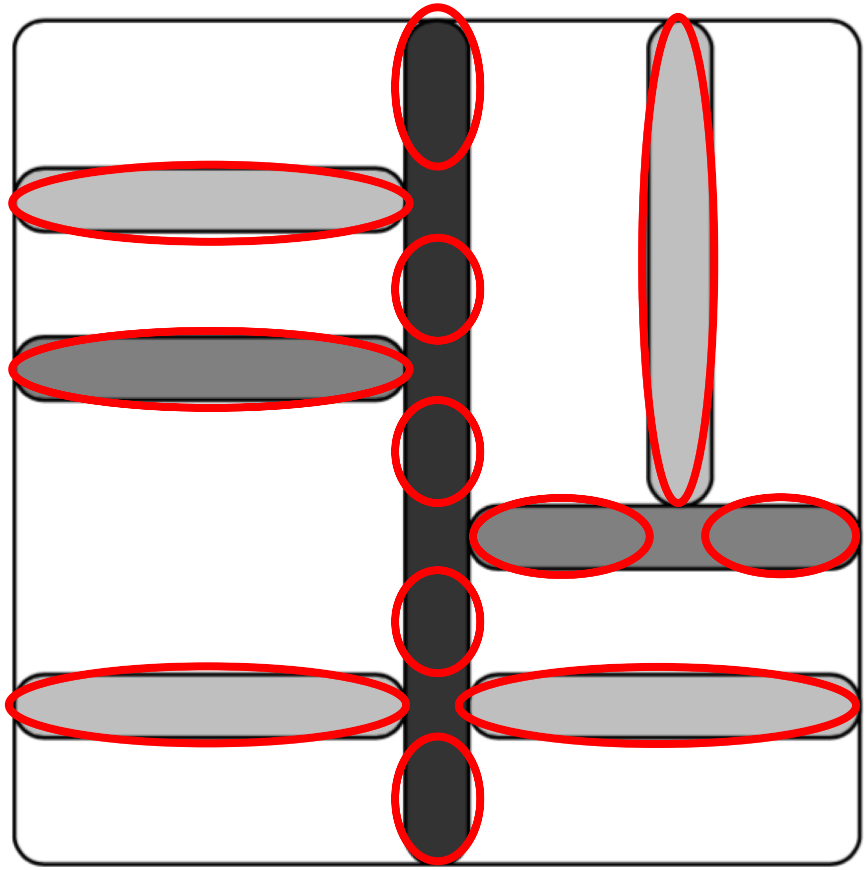
(3) We end up with

$$\begin{bmatrix} C_{ff} & C_{fc} & & \varepsilon \\ C_{cf} & A_{cc} & & A_{cn} \\ & & A_{ww} & A_{wn} \\ \varepsilon & A_{nc} & A_{nw} & A_{nn} \end{bmatrix}$$

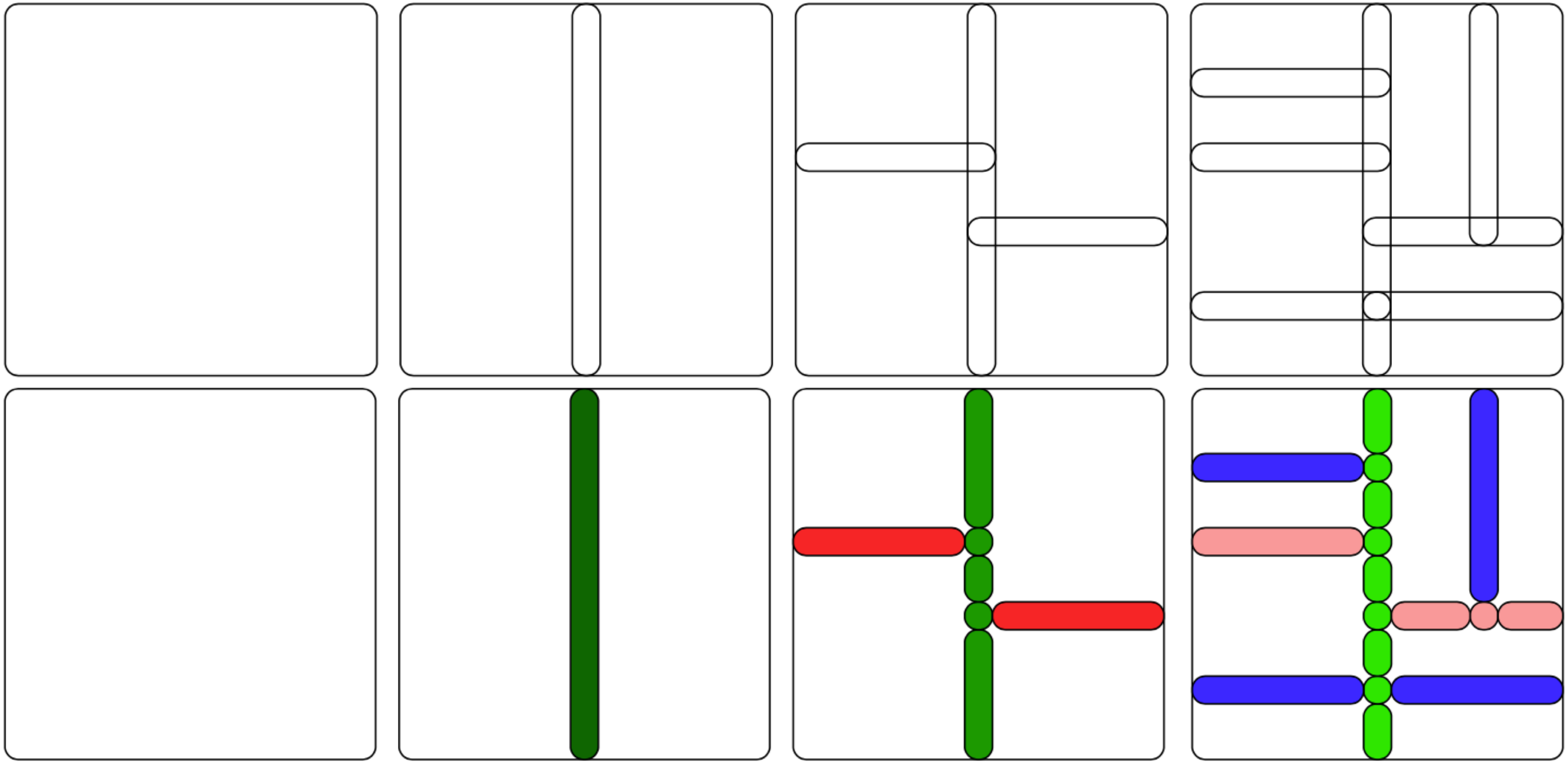


What do we sparsify?

Interfaces between eliminated-interiors



How do we find those interfaces? Coloring

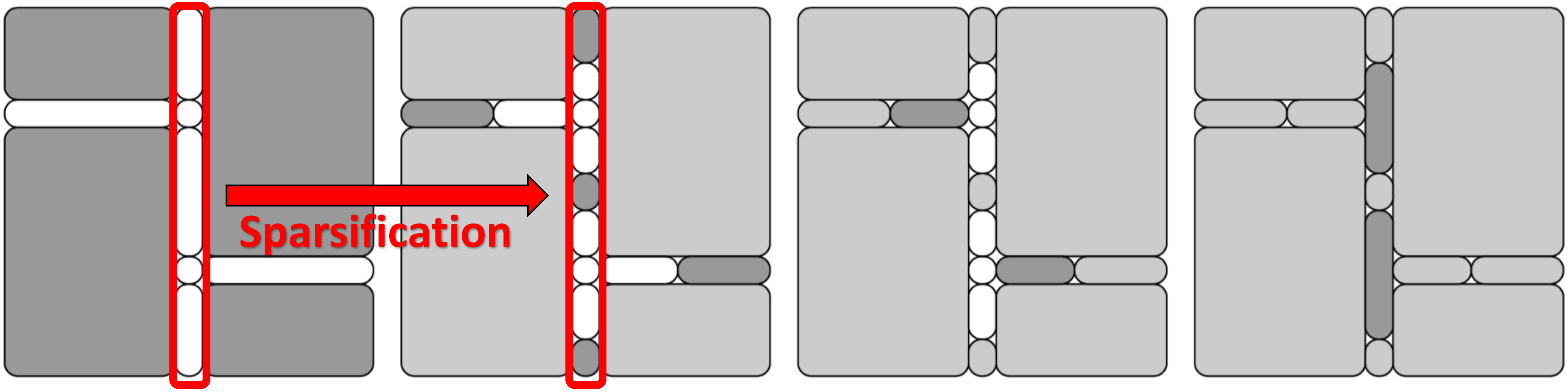


Sparsified Nested Dissection

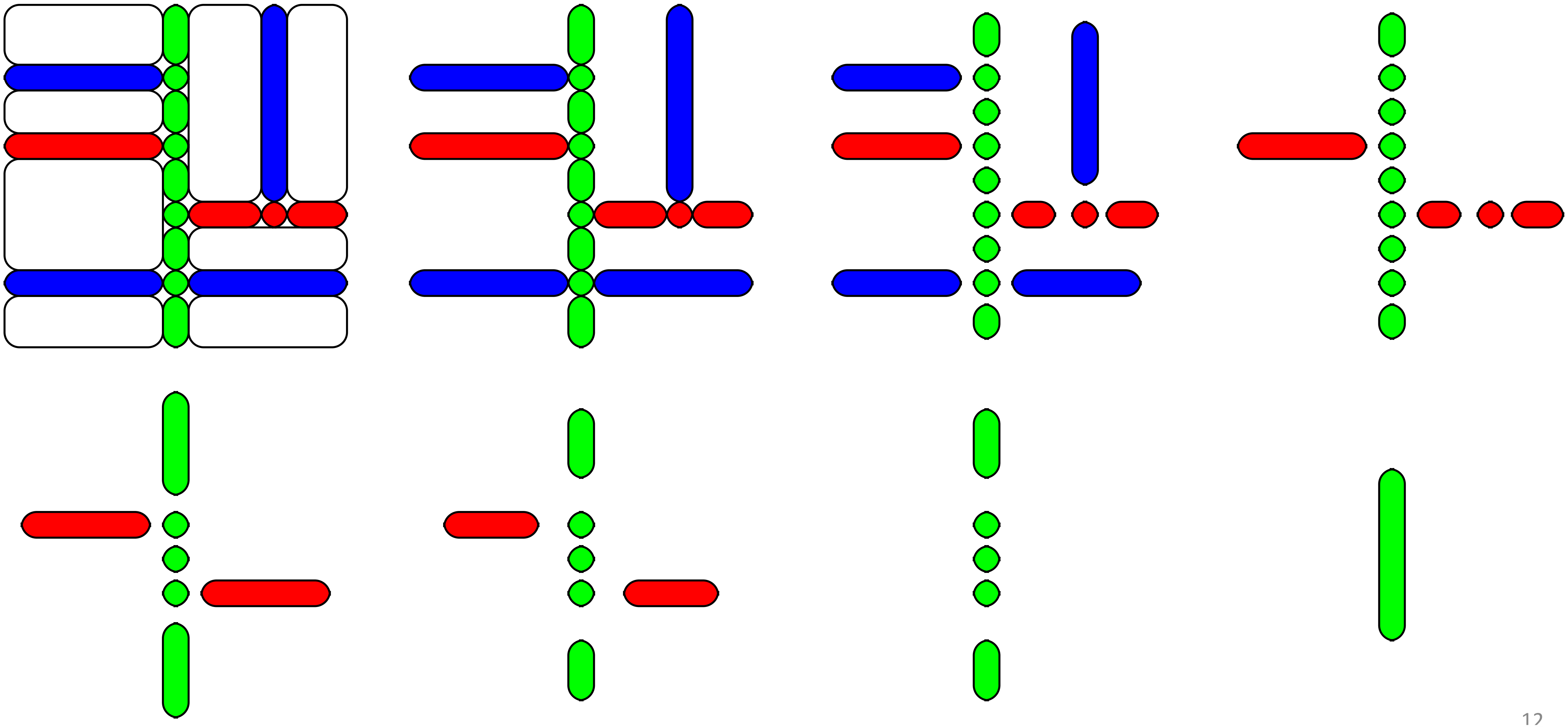
For level ℓ, from leaves to top

Eliminate interiors at level ℓ

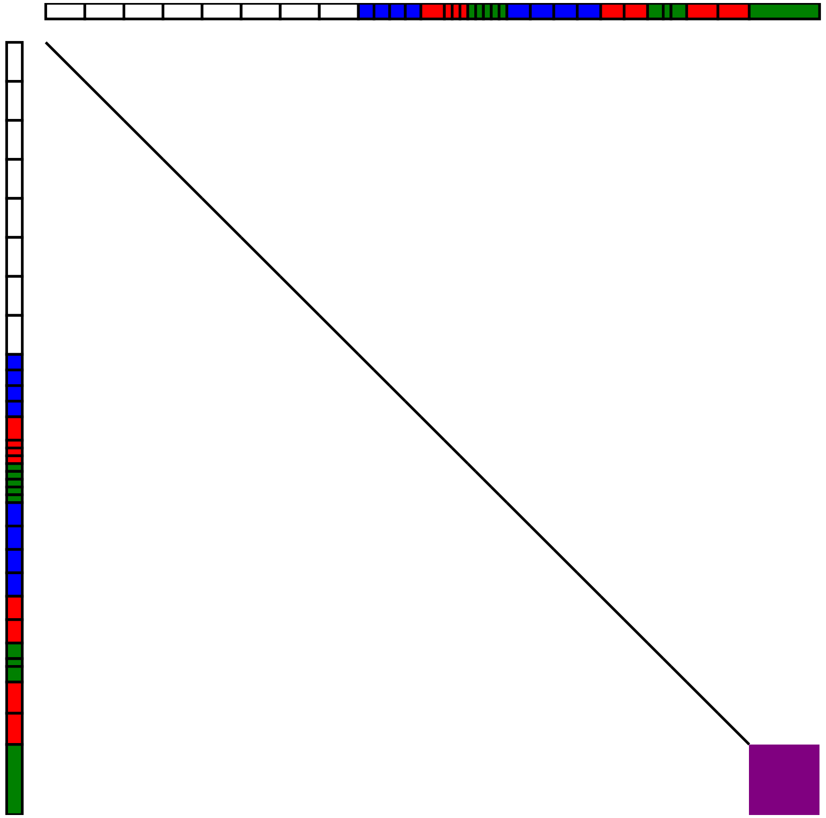
(Scale &) Sparsify interfaces at level ℓ



Sparsified Nested Dissection



Sparsified Nested Dissection



Sparsified Nested Dissection

We effectively build a preconditioner P such that

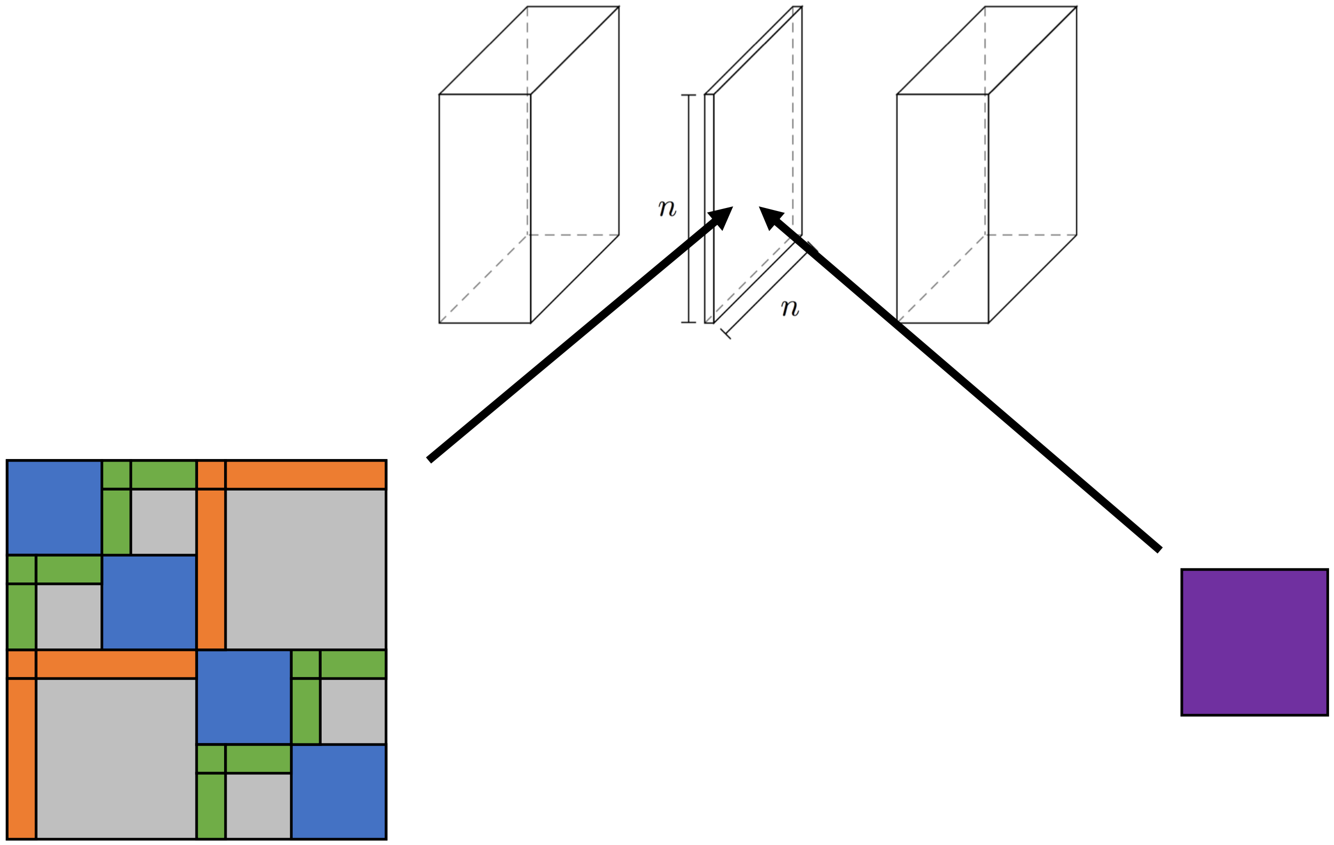
$$P^\top AP \approx I + \varepsilon$$

Where P is a sequence (product) of

- Eliminations
- (Scalings)
- Sparsifications

We then use P as a preconditioner for CG

Different from fast-algebra techniques



Sparsification I & II

$$\begin{bmatrix} \boldsymbol{A_{ss}} & & A_{sn} \\ & A_{ww} & A_{wn} \\ A_{ns} & A_{nw} & A_{nn} \end{bmatrix}$$

$$A_{sn} = \begin{pmatrix} T_{fc} \\ I \end{pmatrix} A_{cn} + \varepsilon$$

OR

$$\begin{bmatrix} \boldsymbol{I} & & A_{sn} \\ & A_{ww} & A_{wn} \\ A_{ns} & A_{nw} & A_{nn} \end{bmatrix}$$

$$A_{sn} = \begin{pmatrix} T_{fc} \\ I \end{pmatrix} A_{cn} + \varepsilon$$

$$A_{sn} = Q_{sc} W_{cn} + \varepsilon$$

Orthogonal, with scaling, stays SPD

$$\begin{bmatrix} \boldsymbol{I} & & A_{sn} \\ & A_{ww} & A_{wn} \\ A_{ns} & A_{nw} & A_{nn} \end{bmatrix}$$

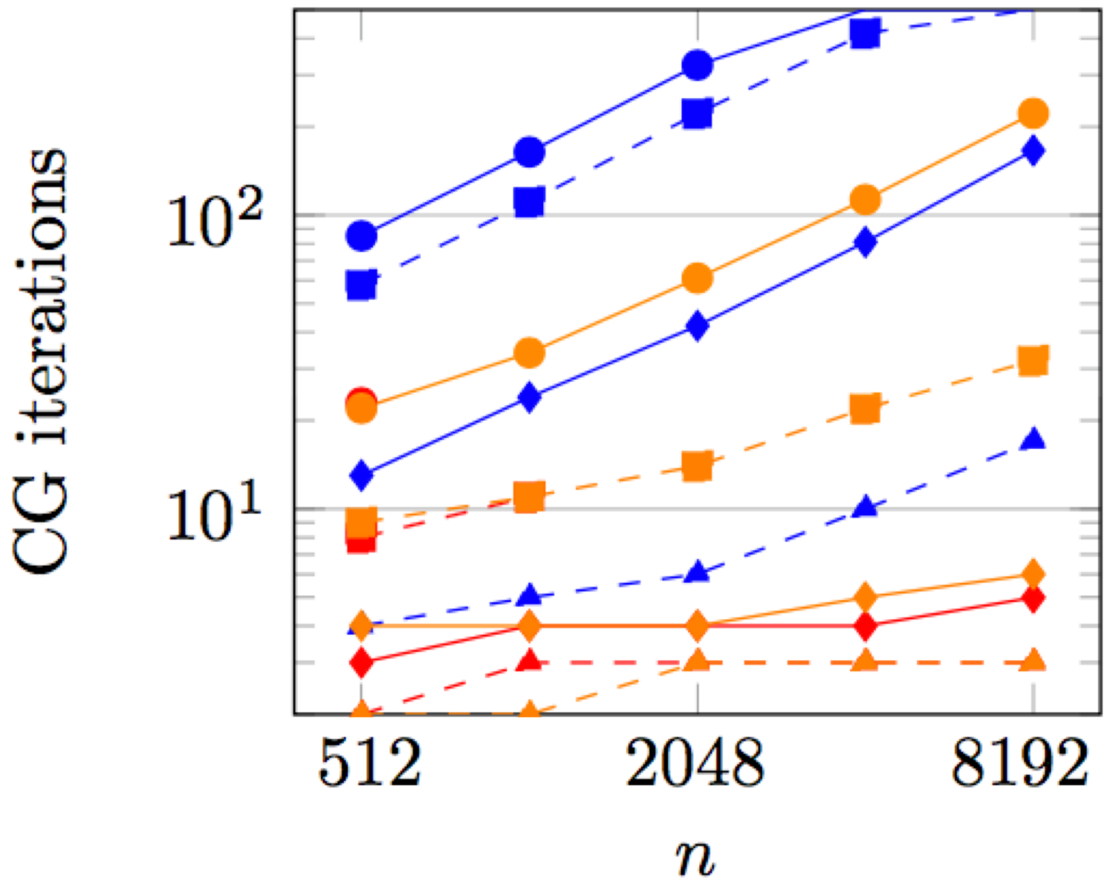
$$A_{sn} = Q_{sc}W_{cn} + \boxed{Q_{sf}W_{fn}}$$

$O(\varepsilon)$

$$\boxed{S_{nn} = A_{nn} - W_{cn}^{\top}W_{cn}} - W_{cf}^{\top}W_{cf}$$

Approximate Schur Complement over (n,n)

Low-Rank Compression: three variants (2D Laplacians)



Interpolative, no scaling

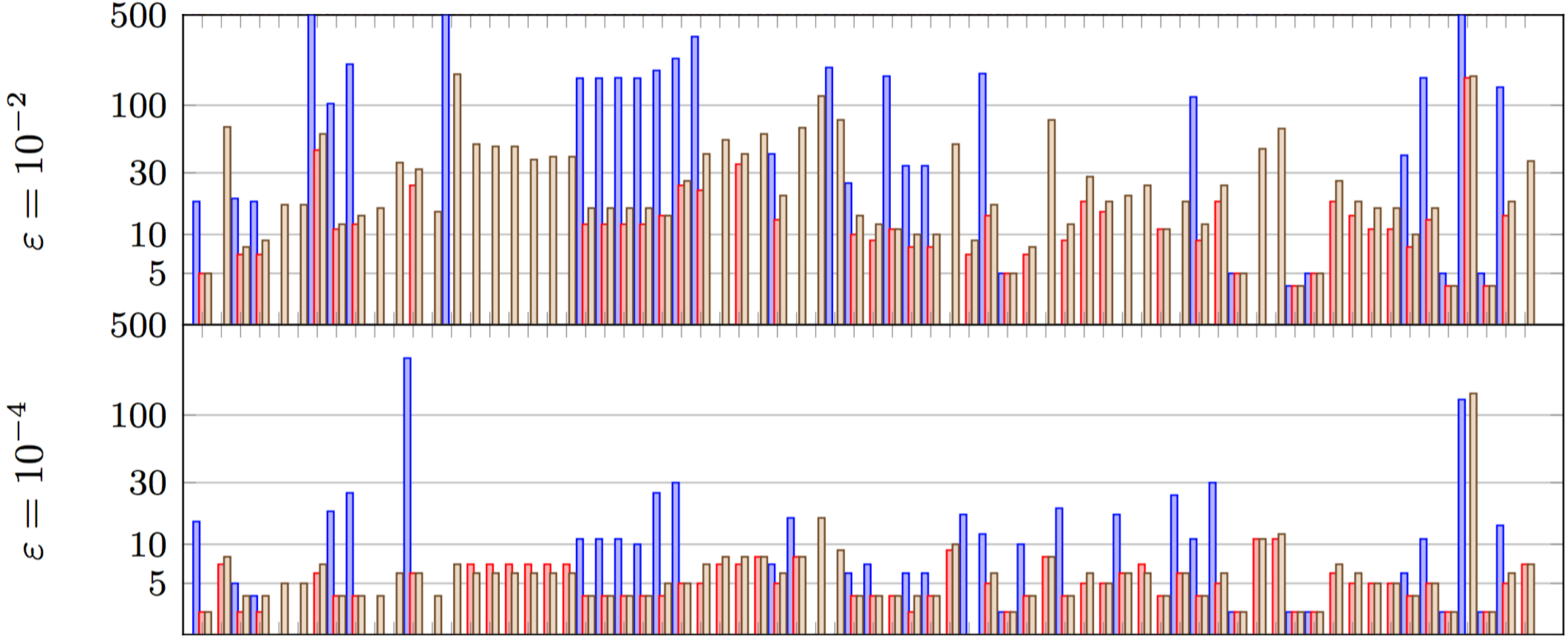
Interpolative, with scaling

Orthogonal, with scaling

$\varepsilon = 10^{-1} \to 10^{-6}$

SPD problems from SuiteSparse

Interpolative, no scaling
Interpolative, with scaling
Orthogonal, with scaling

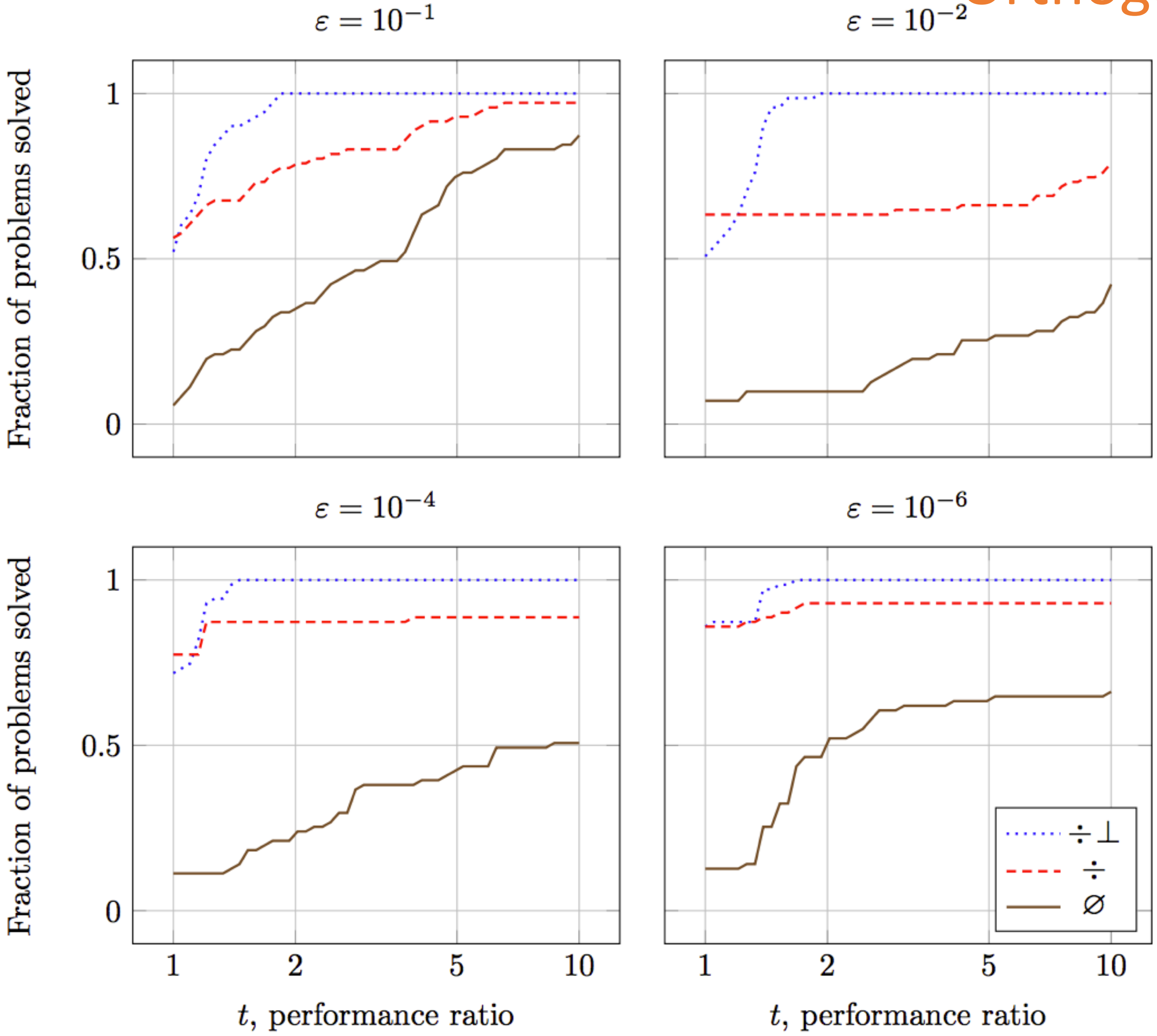


SPD problems from SuiteSparse

Interpolative, no scaling
Interpolative, with scaling
Orthogonal, with scaling

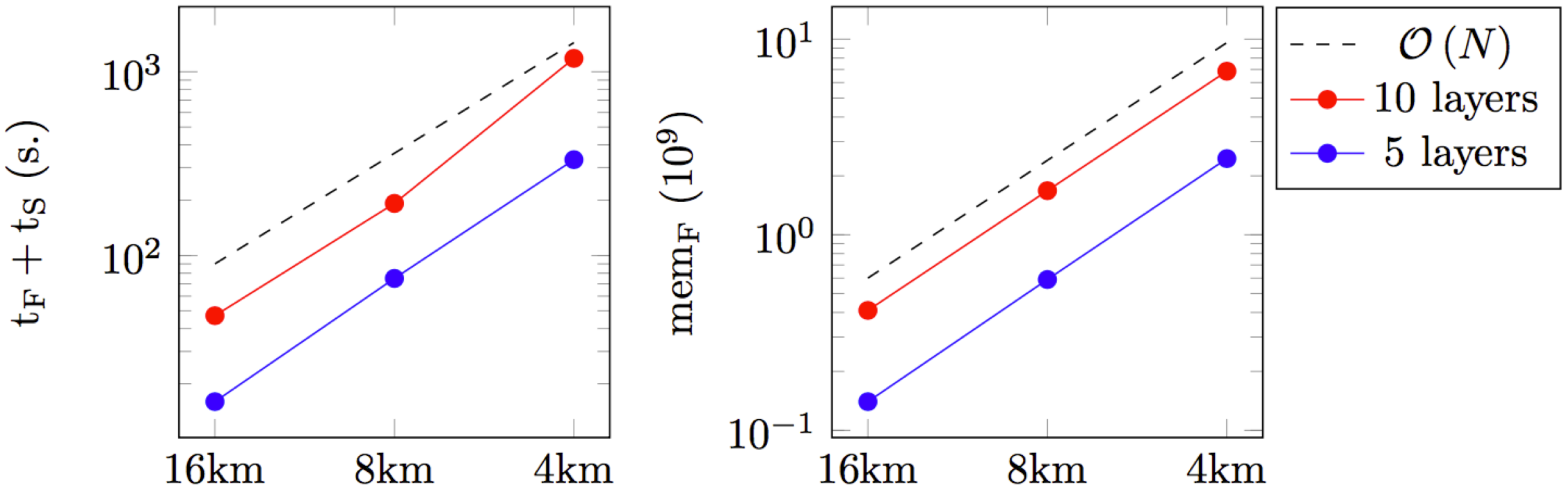
Perf ratio(t) =

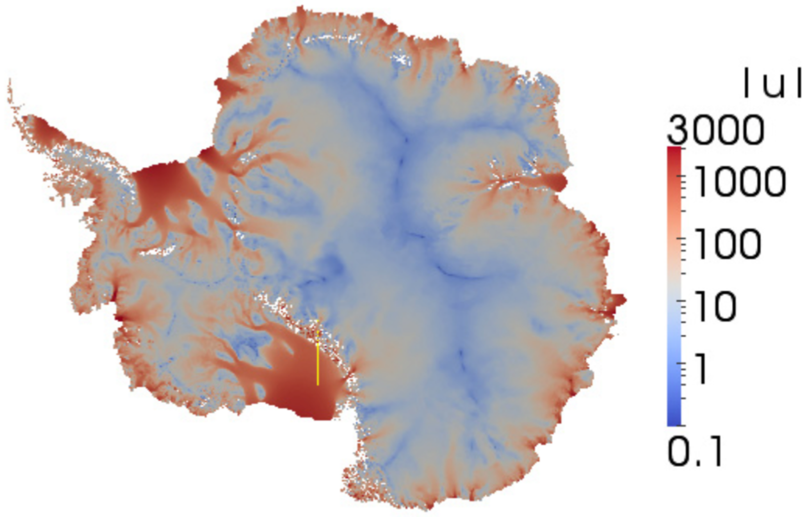
$$\frac{\#\left\{p \in P \mid \frac{CG_{pv}}{CG_p^*} \leq t\right\}}{\#P}$$

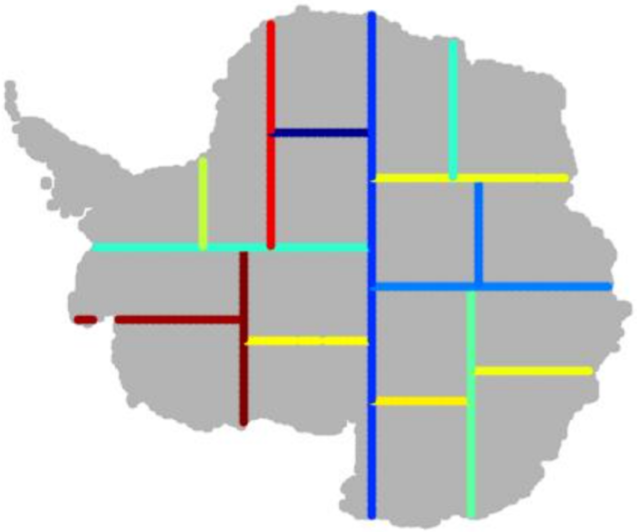


Ice-Sheet modeling $\kappa(A) > 10^{11}$

N	spaND t_F (s.)	t_S (s.)	n_{CG}	size$_{Top}$	mem$_F$ (10^9)	Direct $t_F + t_S$ (s.)
5 layers						
629 544 (16 km)	13	3	7	76	0.14	22
2 521 872 (8 km)	55	20	8	89	0.59	206
10 096 080 (4 km)	217	115	10	100	2.45	1578
10 layers						
1 154 164 (16 km)	39	8	7	136	0.41	90
4 623 432 (8 km)	148	44	8	148	1.68	710
18 509 480 (4 km)	798	384	10	159	6.86	—

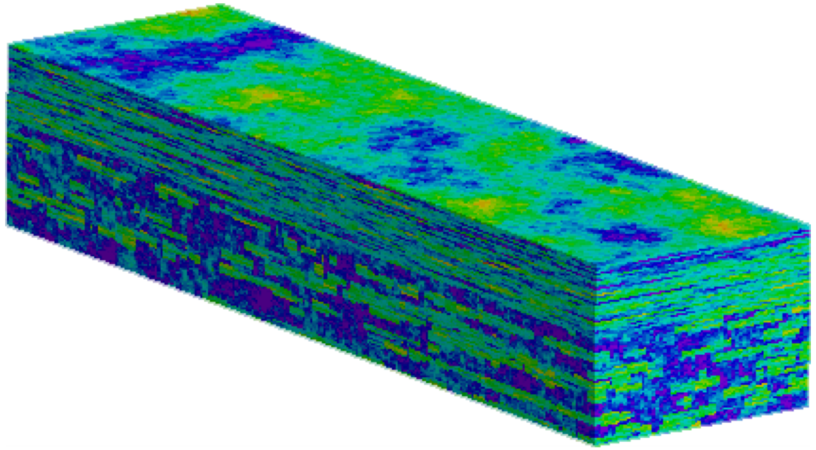


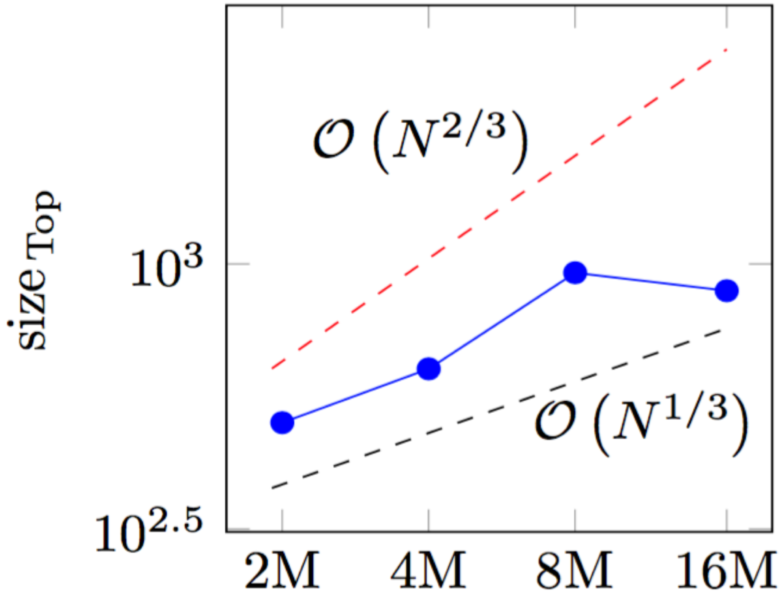


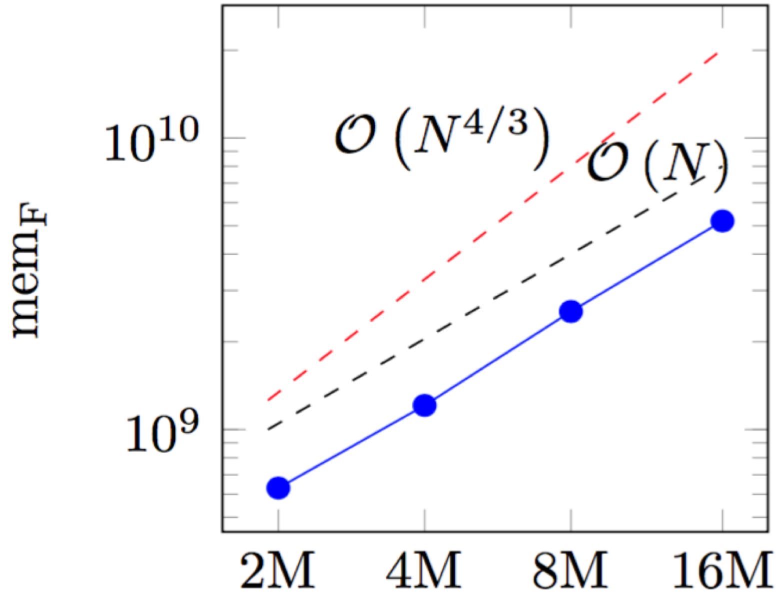


The SPE problem

n	$N = n^3$	spaND t_F (s.)	t_S (s.)	n_{CG}	size $_{Top}$	mem_F (10^9)	Direct. $t_F + t_S$ (s.)
128	2 097 152	61	23	12	502	0.63	686
160	4 096 000	175	46	13	634	1.21	—
200	8 000 000	287	158	16	962	2.54	—
252	16 003 008	963	369	16	890	5.19	—



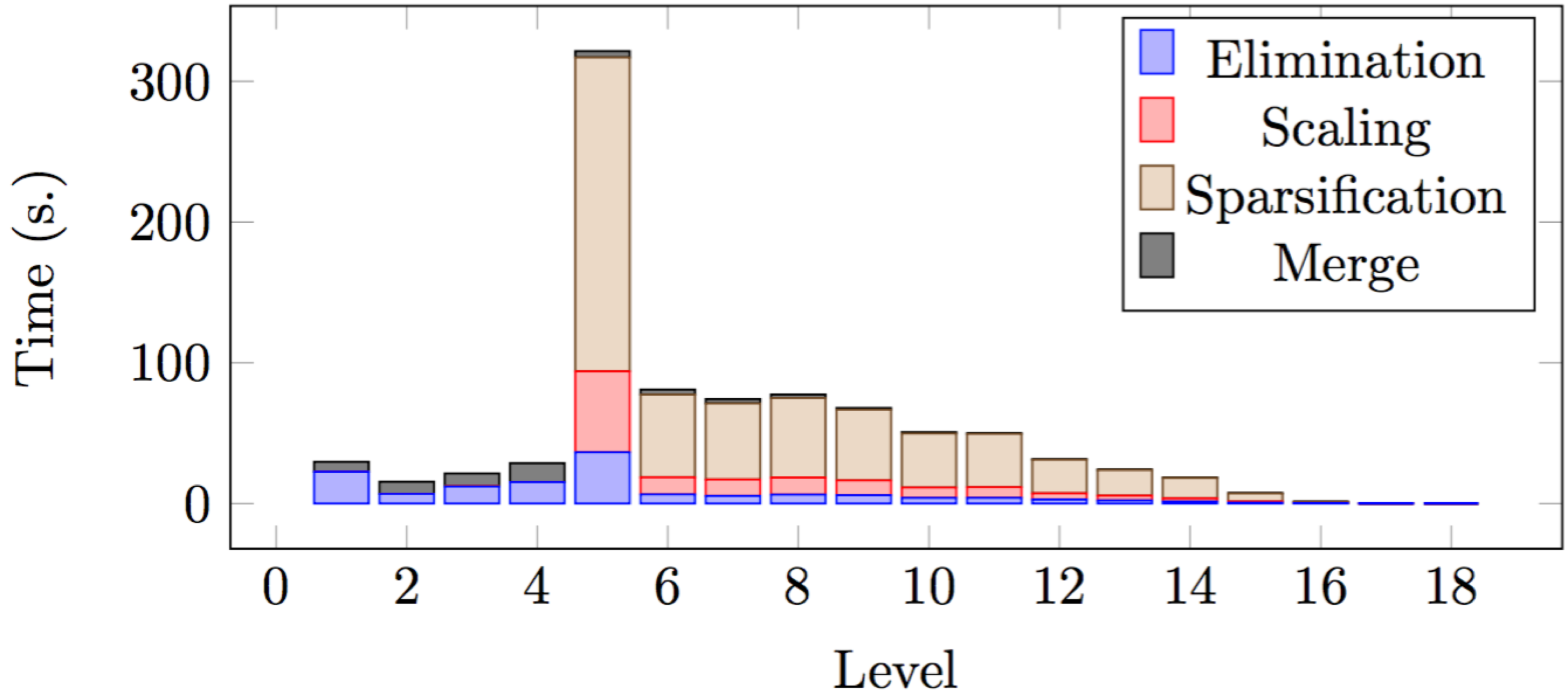




Top separator block would be 32 GB without the sparsification!

Profiling: main cost is RRQR

Optimum is to skip sparsification for levels 1 to 4



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- References:
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